Solutions final test Advanced Calculus (2DBN10)
October 2020

No rights can be derived from these solutions.

1. The characteristic polynomial is $\lambda^3 + \lambda - 2$ with roots $\lambda_1 = 1$, $\lambda_{2,3} = (-1 \pm \sqrt{7}i)/2$. So the general solution to the homogeneous equation is

$$y_h(t) = C_1 e^t + e^{-t/2}\big(C_2 \cos(\sqrt{7}t/2) + C_3 \sin(\sqrt{7}t/2)\big).$$

We find a particular solution to the inhomogeneous problem from the ansatz

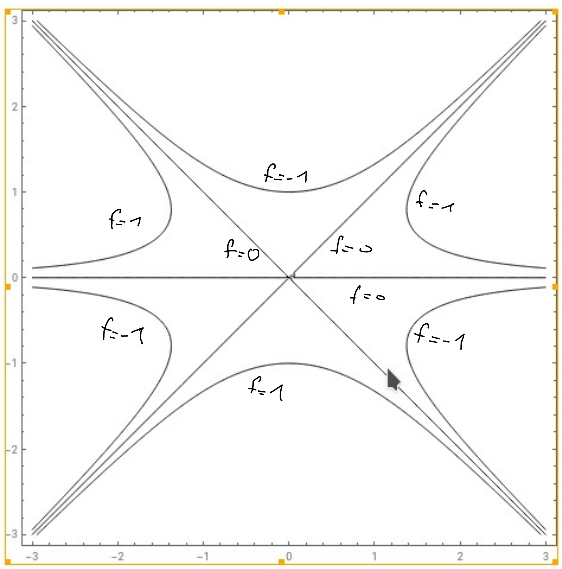
$$y_p(t) = Ate^t$$

and calculating $A = 1/4$. So the general solution to the inhomogeneous equation is

$$y(t) = \frac{1}{4}te^t + C_1 e^t + e^{-t/2}\big(C_2 \cos(\sqrt{7}t/2) + C_3 \sin(\sqrt{7}t/2)\big).$$

(This can also be solved via Laplace transform.)

2.



3. **a)** By the chain rule,

$$\begin{aligned} g'(t) &= \partial_1 f(t,t) + \partial_2 f(t,t), \\ h'(t) &= -\sqrt{2}\sin t\, \partial_1 f(\sqrt{2}\cos t, \sqrt{2}\sin t) + \sqrt{2}\cos t\, \partial_2 f(\sqrt{2}\cos t, \sqrt{2}\sin t). \end{aligned}$$

Setting $t = 1$ in the first equation and $t = \pi/4$ in the second equation yields the linear system

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial_1 f \\ \partial_2 f \end{pmatrix} (1,1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

so $\nabla f(1,1) = (1,2)^\top$.

b) By the chain rule,

$$\begin{aligned}
g''(t) &= \partial_{11}f(t,t) + 2\partial_{12}f(t,t) + \partial_{22}f(t,t),\\
h''(t) &= -\sqrt{2}\cos t\,\partial_1 f(\sqrt{2}\cos t, \sqrt{2}\sin t) - \sqrt{2}\sin t\,\partial_2 f(\sqrt{2}\cos t, \sqrt{2}\sin t)\\
&\quad +2\sin^2 t\,\partial_{11}f(\sqrt{2}\cos t, \sqrt{2}\sin t) - 4\sin t\cos t\,\partial_{12}f(\sqrt{2}\cos t, \sqrt{2}\sin t)\\
&\quad +2\cos^2 t\,\partial_{22}f(\sqrt{2}\cos t, \sqrt{2}\sin t).
\end{aligned}$$

Setting $t = 1$ in the first equation and $t = \pi/4$ in the second equation and using the result from a) yields the linear system

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \partial_{11}f + \partial_{22}f \\ \partial_{12}f \end{pmatrix} (1,1) = \begin{pmatrix} 0 \\ 4 \end{pmatrix},$$

so $\partial_{12}f(1,1) = -1$.

4. Step 1: Critical points (x, y) of f satisfy

$$\nabla f(x,y) = \frac{1}{1+4x^2+y^2} \begin{pmatrix} 2 - \frac{8x(2x+y)}{1+4x^2+y^2} \\ 1 - \frac{2y(2x+y)}{1+4x^2+y^2} \end{pmatrix} = 0.$$

This implies $2x + y \neq 0$ and then $y = 2x$, and from this $16x^2 = 1 + 8x^2$, hence $(x,y) = \pm(\frac{1}{4}\sqrt{2}, \frac{1}{2}\sqrt{2})$. These critical points are in D and have function values $\pm\frac{1}{2}\sqrt{2}$.

Step 2: The boundary of D is an ellipse that can be parameterized by $x = \cos t$, $y = 2\sin t$. Extrema on the boundary can now be found by finding the extrema of the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$g(t) = f(\cos t, 2\sin t) = \frac{2}{5}(\cos t + \sin t).$$

By 1D Calculus we find that the extreme values at the boundary are $\pm\frac{2}{5}\sqrt{2}$ (at the points $\pm(\frac{1}{2}\sqrt{2}, \sqrt{2})$).

Therefore, the global maximum and minimum of f on D are $\pm\frac{1}{2}\sqrt{2}$.

5. The coefficient matrix has the eigenvalues -2 and 1 with corresponding eigenvectors $(-2,1)^\top$ and $(-3,2)^\top$. So the general solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{t},$$

As $y_1(t)$, $y_2(t) \to 0$ for $t \to \infty$ we find $C_2 = 0$, and then from $y_1(0) = 1$ we find $C_2 = -1/2$.

6. We have that

$$f(u,v) = u - 1 + 2v - (u-1)^2 + (u-1)v - 2v^2 + O(|(u-1,v)|^3).$$

Using this together with the standard 1D expansions

$$e^x = 1 + x + x^2/2 + O(x^3), \qquad \frac{y}{1-y} = y + y^2 + O(y^3)$$

we find

$$g(x,y) = x + 2y - \frac{x^2}{2} + xy + O(|(x,y)|^3),$$

i.e. the Taylor polynomial of g around $(0,0)$ is given by

$$p(x,y) = x + 2y - \frac{x^2}{2} + xy.$$

(The problem can also be solved by first identifying the partial derivatives of f up to order 2, calculating the partial derivatives of g via the chain rule, and then applying the standard formula.)

7. The Lagrange equations are

$$\nabla f(x,y,z) = \frac{1}{z+2}\begin{pmatrix} 1 \\ 1 \\ -\frac{x+y}{z+2} \end{pmatrix} = 2\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This implies $x = y \neq 0$ and $\lambda = \frac{1}{2x(z+2)}$. Inserting this in the last equation yields

$$-\frac{2x}{(z+2)^2} = \frac{z}{x(z+2)},$$

and together with the restriction $x^2 + y^2 + z^2 = 2x^2 + z^2 = 1$ we find from this

$$-2x^2 = z(z+2) = z^2 - 1,$$

so $z = -1/2$, $x = y = \pm\frac{1}{4}\sqrt{6}$ with corresponding function values $\pm\frac{1}{3}\sqrt{6}$. As there are only two solutions to the Lagrange equations, these correspond to the global maximum and minimum of f on S.

8.

$$\iiint_K z\,dV = \int_0^1 \int_0^{2-z} \int_0^{2-y-z} z\,dxdydz = \frac{11}{24}.$$