Solutions final test Advanced Calculus (2DBN10)
January 2021

No rights can be derived from these solutions.

1. a) Let $Y_{1,2}$ denote the Laplace transforms of $y_{1,2}$. By calculation laws for the Laplace transform,

$$\begin{pmatrix} s-1 & -2 \\ -2 & s-1 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} F_1(s)+a_1 \\ F_2(s)+a_2 \end{pmatrix}.$$

Solving the linear system yields

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \frac{1}{(s-1)^2-4} \begin{pmatrix} (s-1)(F_1(s)+a_1)+2(F_2(s)+a_2) \\ 2(F_1(s)+a_1)+(s-1)(F_2(s)+a_2) \end{pmatrix}.$$

b) We have $F_1(s) = -F_2(s) = \frac{1}{s+1}$ and therefore

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \frac{1}{(s-1)^2-4} \begin{pmatrix} (s-1)\left(\frac{1}{s+1}+1\right)-\frac{2}{s+1} \\ 2\left(\frac{1}{s+1}+1\right)-\frac{s-1}{s+1} \end{pmatrix} = \frac{1}{(s+1)^2(s-3)} \begin{pmatrix} s^2+s-4 \\ s+5 \end{pmatrix}$$

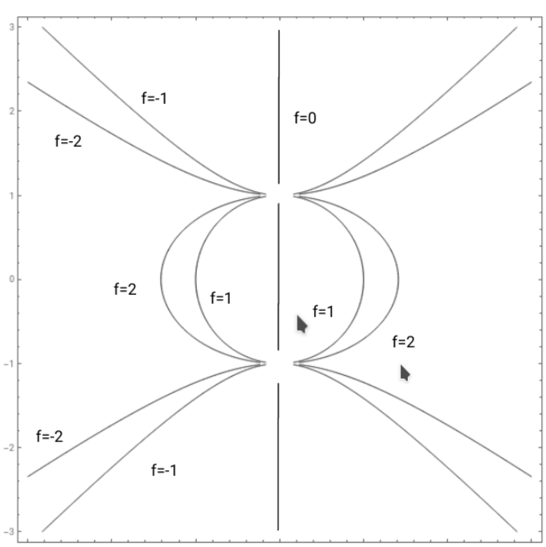
and after partial fraction decomposition

$$\begin{aligned} Y_1(s) &= \frac{1}{2}\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s-3}, \\ Y_2(s) &= -\frac{1}{2}\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s-3}, \end{aligned}$$

so by inverse Laplace transform

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{-t} + te^{-t} + \frac{1}{2}e^{3t}, \\ y_2(t) &= -\frac{1}{2}e^{-t} - te^{-t} + \frac{1}{2}e^{3t}. \end{aligned}$$

2. a)



(All level curves approach the points $(0,1)$ and $(0,-1)$. However, they do not intersect as these points are not part of the domain of definition.)

b) We have $\partial_x f(1,2) = -2/3$, $\partial_y f(1,2) = 4/9$, so the equation is

$$z = -\frac{1}{3} - \frac{2}{3}(x-1) + \frac{4}{9}(y-2).$$

c)

$$-\frac{2}{3}(x-1) + \frac{4}{9}(y-2) = 0, \text{ or } -3x + 2y = 1.$$

3. **a)** Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be given by

$$F(x,y,u,v) = \Phi(x,y) - (u,v)^\top = \begin{pmatrix} x^3 - y^3 - u \\ xy - v \end{pmatrix}.$$

So $\Phi(x,y) = (u,v)^\top$ if and only if $F(x,y,u,v) = 0$. We apply the Implicit Function theorem to this equation near the point $(1,1,\Phi(1,1)) = (1,1,0,1)$. We have $F(1,1,0,1) = 0$ and

$$D_{(x,y)}F(1,1,0,1) = D\Phi(1,1) = \begin{pmatrix} 3x^2 & -3y^2 \\ y & x \end{pmatrix}\Bigg|_{(x,y)=(1,1)} = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}$$

is regular. So Φ is locally invertible near $(1,1)$.

b) As

$$\Phi^{-1}(\Phi(x,y)) = (x,y)^\top$$

for (x,y) near $(1,1)$, we have by the chain rule

$$D(\Phi^{-1})(\Phi(x,y))D\Phi(x,y) = I$$

and, in particular, for $(x,y) = (1,1)$,

$$D(\Phi^{-1})(0,1) = (D\Phi(1,1))^{-1} = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}.$$

4. Let $\Phi : T_1 \longrightarrow T_a$ be given by $\Phi(x,y,z) = (ax, ay, az)$. This obviously is an injective, differentiable map with $D\Phi(x) = aI$ and $\det D\Phi(x) = a^3$, so

$$\int_{T_a} f\,dV = \int_{T_1} (f \circ \Phi)|\det D\Phi|\,dV = a^3 \int_{T_1} f(ax)\,dx = a^3 \int_{T_1} a^2 f(x)\,dx = a^5.$$

5. Any solution y can be written as

$$y(t) = \sum_{k=1}^{2021} C_k e^{\lambda_k t}$$

with constants $C_k \in \mathbb{C}$ and λ_k the 2021 solutions to the characteristic equation $\lambda^{2021} - 1 = 0$. All these satisfy $|\lambda_k^{2021}| = |\lambda_k|^{2021} = 1$ and therefore $|\lambda_k| = 1$. Consequently, for $t \geq 0$,

$$|e^{\lambda_k t}| = e^{\operatorname{Re}\lambda_k t} \leq e^t$$

and so

$$|y(t)| \leq \sum_{k=1}^{2021} |C_k||e^{\lambda_k t}| \leq Ce^t, \qquad C := \sum_{k=1}^{2021} |C_k|.$$

6. By the chain rule,

$$\begin{aligned}\partial_x \Phi(x,y) &= 2x\partial_1 f(x^2+y^3,xy) + y\partial_2 f(x^2+y^3,xy) = 1,\\ \partial_y \Phi(x,y) &= 3y^2\partial_1 f(x^2+y^3,xy) + x\partial_2 f(x^2+y^3,xy) = 2,\end{aligned}$$

and for $(x,y)=(1,1)$

$$\begin{aligned}\partial_x \Phi(1,1) &= 2\partial_1 f(2,1) + \partial_2 f(2,1) = 1,\\ \partial_y \Phi(1,1) &= 3\partial_1 f(2,1) + \partial_2 f(2,1) = 2.\end{aligned}$$

Solving this linear system for $\partial_1 f(2,1)$, $\partial_2 f(2,1)$ yields

$$\nabla f(2,1) = (1,-1)^\top.$$

7. The Lagrange equations are

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} \frac{a}{x^2} \\ \frac{b}{y^2} \\ \frac{c}{z^2} \end{pmatrix},$$

so

$$\lambda = \frac{x^2yz}{a} = \frac{xy^2z}{b} = \frac{xyz^2}{c},$$

hence $x/a = y/b = z/c$, and using the restriction we get

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{1}{3}.$$

So the minimum is taken for $x=3a$, $y=3b$, $z=3c$, and its value is $27abc$.

8. Cylindrical coordinates:

$$\iiint_K z\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} zr\,dzdrd\theta = \frac{5}{24}\pi.$$