Solutions final test Advanced Calculus (2DBN10)
November 2021

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1. The corresponding homogeneous equation has characteristic polynomial $\lambda \mapsto \lambda^3 - 1$ with roots $\lambda_1 = 1$, $\lambda_{2,3} = (-1 \pm \sqrt{3}i)/2$. Therefore the general solution to the homogeneous equation is

$$y_h(t) = C_1 e^t + e^{-t/2}\big(C_2 \cos(\sqrt{3}t/2) + C_3 \sin(\sqrt{3}t/2)\big).$$

To find a particular solution we use the ansatz $y_p = Ate^t$ and find $y_p''' - y_p = 3Ae^t$, hence $A = 1/3$. So the general solution is

$$y(t) = (C_1 + t/3)e^t + e^{-t/2}\big(C_2 \cos(\sqrt{3}t/2) + C_3 \sin(\sqrt{3}t/2)\big).$$

2. **a)** Let $\underline{u}$ and $\underline{v}$ be two solutions to the given system. Their difference $\underline{u} - \underline{v}$ solves the corresponding homogeneous system. The system matrix has eigenvalues $\lambda_{1,2} = -1 \pm 2i$, so there are constants $C_{1,2}$ such that

$$\underline{u}(t) - \underline{v}(t) = C_1 \operatorname{Re}\left(e^{(-1+2i)t}\underline{w}\right) + C_2 \operatorname{Im}\left(e^{(-1+2i)t}\underline{w}\right),$$

where $\underline{w} \in \mathbb{R}^2$ is an eigenvector of the system matrix belonging to the eigenvalue $-1+2i$. Thus,

$$|\underline{u}(t) - \underline{v}(t)|_{\mathbb{R}^2} \le C|e^{(-1+2i)t}| = Ce^{-t} \overset{t\to\infty}{\longrightarrow} 0,$$

which implies the result.

b) Define the functions $h, m : [0,\infty) \longrightarrow \mathbb{R}$ by

$$h(t) = \int_0^t f(\tau)\, d\tau, \qquad m(t) = h(2t).$$

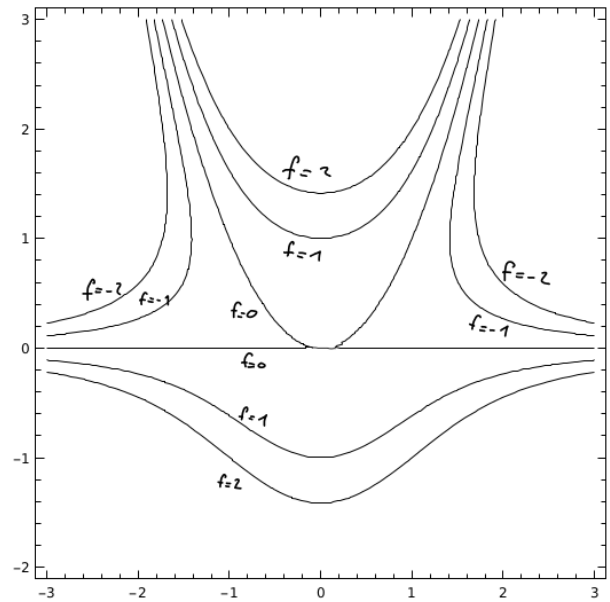
Let $H, M : (0,\infty) \longrightarrow \mathbb{R}$ denote their respective Laplace transforms. According to the calculation rules,

$$H(s) = \frac{F(s)}{s}, \qquad M(s) = \frac{1}{2} H\left(\frac{s}{2}\right) = \frac{1}{s} F\left(\frac{s}{2}\right),$$

and from $g(t) = m(t) - h(t)$ we get

$$G(s) = \frac{1}{s}\left(F\left(\frac{s}{2}\right) - F(s)\right), \qquad s > 0.$$

3. **a)**



b) $z = f(1,2) + \partial_x f(1,2)(x-1) + \partial_y f(1,2)(y-2) = 2 - 4(x-1) + 3(y-2)$

c) $-4(x-1) + 3(y-2) = 0$, or equivalently $y = \frac{4}{3}x + \frac{2}{3}$

4. Using standard expansions, we get

$$\frac{y}{2-x} = \frac{y}{1-(x-1)} = y + y(x-1) + y(x-1)^2 + O(|(x-1,y)|^4),$$

$$\left(\frac{y}{2-x}\right)^3 = y^3 + O(|(x-1,y)|^4).$$

and further, using $\sin z = z - z^3/6 + O(z^5)$,

$$\sin\left(\frac{y}{2-x}\right) = y + y(x-1) + y(x-1)^2 - y^3/6 + O(|(x-1,y)|^4).$$

So the Taylor polynomial is given by

$$T_{f,(1,0),2}(x,y) = y + y(x-1) + y(x-1)^2 - y^3/6.$$

(This can alternatively obtained by using the standard formula.)

5. The function f has no critical points in D, as its only critical point is the origin.

The boundary of D consists of two components that can be parameterized by

$$(I): \quad y = 3 - x, \qquad 1 \le x \le 2,$$
$$(II): \quad y = 2/x, \qquad 1 \le x \le 2.$$

(I): The extremal values on this component are found by calculating the extrema of the function g given by

$$g(x) = f(x, 3-x) = 3x^2 - 12x + 18$$

on the interval $[1,2]$. The minimal value is $f(2,1) = 6$ and the maximal value is $f(1,2) = 9$.

(II): The extremal values on this component are found by calculating the extrema of the function h given by

$$h(x) = f(x, 2/x) = x^2 + 8/x^2$$

on the interval $[1,2]$. The minimal value is $f(\sqrt[4]{8},\sqrt[4]{2}) = 4\sqrt{2}$ and the maximal value is $f(1,2) = 9$.

Summarizing, the global minimum of f on D is $f(\sqrt[4]{8},\sqrt[4]{2}) = 4\sqrt{2}$, and the global maximum is $f(1,2) = 9$.

6. The Lagrange equations are

$$a = \lambda yz, \quad b = \lambda xz, \quad c = \lambda xy,$$

hence

$$\lambda xyz = ax = by = cz.$$

This implies

$$y = \frac{ax}{b}, \quad z = \frac{ax}{c}$$

and because of $xyz = 1$

$$x = \sqrt[3]{\frac{bc}{a^2}}$$

and similarly

$$y = \sqrt[3]{\frac{ac}{b^2}}, \quad z = \sqrt[3]{\frac{ab}{c^2}}.$$

With these values, we find the minimum

$$ax + by + cz = 3\sqrt[3]{abc}.$$

7. For the Jacobian of Φ we get

$$\det D\Phi(u,v) = \det\left(\frac{1}{(u^2+v^2)^2}\begin{pmatrix} 2(v^2-u^2) & -4uv \\ 2uv & v^2-u^2 \end{pmatrix}\right) = 2\frac{(v^2-u^2)^2+4u^2v^2}{(u^2+v^2)^4} = \frac{2}{(u^2+v^2)^2}.$$

For the area of $\Phi(D)$ we get by applying the change-of-variable theorem and then using polar coordinates

$$\iint_{\Phi(D)} 1\,dA = \iint_D |\det D\Phi|\,dA = \iint_D \frac{2}{(u^2+v^2)^2}\,dudv = 2\int_0^{2\pi}\int_1^2 r^{-3}\,drd\theta = \frac{3}{2}\pi.$$

8. Cylindrical coordinates:

$$\iiint_K z\,dV = \int_0^{2\pi}\int_0^1\int_r^{2-r} zr\,dzdrd\theta = \frac{2}{3}\pi.$$