Solutions final test Advanced Calculus (2DBN10)
January 2022

No rights can be derived from these solutions.

1. The eigenvalues of the system matrix are

$$\lambda_{1,2} = \frac{a+1}{2} \pm \sqrt{\left(\frac{a+1}{2}\right)^2 - a - 5}.$$

The system has nonzero periodic solutions if and only if these are purely imaginary, i.e. if $a = -1$. In this case, the eigenvalues are $\pm 2i$, corresponding eigenvectors are $(5, -1 \pm 2i)$, the complex general solution is

$$y(t) = C_1 \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} e^{2it} + C_2 \begin{pmatrix} 5 \\ -1-2i \end{pmatrix} e^{-2it}, \qquad C_{1,2} \in \mathbb{C},$$

and the real general solution is

$$y(t) = D_1 \begin{pmatrix} 5\cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{pmatrix} + D_2 \begin{pmatrix} 5\sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{pmatrix}, \qquad D_{1,2} \in \mathbb{R}.$$

2. The corresponding homogeneous problem has the solution

$$y_h(t) = C_1 \cos t + C_2 \sin(t).$$

Accordingly, we use the "variation of coefficients" ansatz

$$y_p(t) = C_1(t)\cos(t) + C_2(t)\sin(t).$$

With the additional demand

$$C_1'(t)\cos(t) + C_2'(t)\sin(t) = 0$$

we get the linear system

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\cos t} \end{pmatrix}$$

with solution

$$C_1'(t) = -\tan t, \quad C_2'(t) = 1,$$

so (with integration constants omitted)

$$C_1(t) = \ln(\cos t), \quad C_2(t) = t,$$

and

$$y_p(t) = \cos t \ln(\cos t) + t \sin t.$$

3. **a)** The level curves for function value $C \neq 0$ has the equation

$$\frac{4x^2}{y} + y = C,$$

which is equivalent to

$$4x^2 + \left(y - \frac{C}{2}\right)^2 = \frac{C^2}{4}.$$

These curves are ellipses with center $(0, C/2)$ and half axes $|C|/4$ and $|C|/2$ in x- and y-direcction, respectively. (with the point $(0,0)$ removed, as it is not in the domain of definition of f).

b) $z = 5 + 8(x-1) - 3(y-1)$

c) $8(x-1) - 3(y-1) = 0$

4. For $\nabla v(0,0)$ to be determined by $\nabla u(0,0)$,it is necessary that $F(0,0) = (0,0)$. Then, by the chain rule,

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \partial_1 F_1 & \partial_1 F_2 \\ \partial_2 F_1 & \partial_2 F_2 \end{pmatrix} (0,0) \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so for example $F(x,y) = (x-y, x+y)$.

5. The gradient of f is

$$\nabla f(x,y) = \begin{pmatrix} -\frac{1}{x^2} + \frac{9}{(4-x+y)^2} \\ \frac{4}{y^2} - \frac{9}{(4-x+y)^2} \end{pmatrix},$$

so in critical points, we have $1/x^2 = 4/y^2$, or $y = \pm 2x$.

If $y = 2x$ then $1/x^2 = 9/(4+x)^2$ and we obtain the critical points $(x_1, y_1) = (2,4)$ and $(x_2, y_2) = (-1,-2)$.

If $y = -2x$ then $1/x^2 = 9/(4-3x)^2$ and we obtain the critical point $(x_3, y_3) = (2/3, -4/3)$.

The matrix of second derivatives of f is

$$\begin{pmatrix} 2/x^3 & 0 \\ 0 & -8/y^3 \end{pmatrix} + \frac{18}{(4-x+y)^3} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The second derivative test yields that (x_1, y_1) and (x_2, y_2) are saddle points while (x_3, y_3) is a (strict local) minimum.

6. As $z > 0$ on L, the point with minimal distance to the (x,y)-plane is the point with minimal z-coordinate. Hence we have to minimize the function f given by $f(x,y,z) = z$ under the conditions

$$g_1(x,y,z) := z(x^2+y^2) - 1 = 0, \quad g_2(x,y,z) = xyz^2 = 1, \quad x > 0,\ y > 0,\ z > 0.$$

The corresponding Lagrange equations are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2xz \\ 2yz \\ x^2+y^2 \end{pmatrix} + \mu \begin{pmatrix} yz^2 \\ xz^2 \\ 2xyz \end{pmatrix} = 0.$$

A point (x,y,z) yields a solution if and only if

$$\det \begin{pmatrix} 0 & 2xz & yz^2 \\ 0 & 2yz & xz^2 \\ 1 & x^2+y^2 & 2xyz \end{pmatrix} = 2z^3(x^2-y^2) = 0,$$

hence $x = y = 1/2$ and $z = 2$. (The Lagrange equations can also be solved in different ways.)

7. **a)** If such functions exist, then necessarily $\xi(0) = \eta(0) = 1$. Let $F := \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$F(t,x,y) := \begin{pmatrix} t + e^{ty} - x \\ t + e^{2tx} - y \end{pmatrix}.$$

A pair of functions $(\xi, \eta) : I \longrightarrow \mathbb{R}^2$ solves the given equations if an only if $F(t, \xi(t), \eta(t)) = 0$. The function F is differentiable, and we have $F(0,1,1) = (0,0)^\top$ and

$$DF(t,x,y) = \begin{pmatrix} -1 & te^{ty} \\ 2te^{2tx} & -1 \end{pmatrix}, \quad DF(0,1,1) = -I,$$

so that $DF(0,1,1)$ is regular. The existence of the interval I and the functions ξ and η with the demanded properties follows now from the Implicit Function theorem.

b) Differentiating the given equations yields

$$\xi'(t) = 1 + (\eta(t) + t\eta'(t))e^{t\eta(t)},$$

hence $\xi'(0) = 2$, similarly $\eta'(0) = 3$, and

$$\xi''(t) = (2\eta'(t) + t\eta''(t))e^{t\eta(t)} + (\eta(t) + t\eta'(t))^2 e^{t\eta(t)}$$

hence $\xi''(0) = 7$.

8.

$$\begin{aligned}\iiint_K x\,dV &= \int_0^1 \int_0^1 \int_0^{\min(x^2,y^2)} x\,dzdydx \\ &= \int_0^1 x \int_0^x \int_0^{y^2} dzdydx + \int_0^1 x \int_x^1 \int_0^{x^2} dzdydx = \frac{1}{15} + \frac{1}{20} = \frac{7}{60}.\end{aligned}$$

Different integration orders with corresponding different integration limits are possible as well, e.g.

$$\int_0^1 \int_{\sqrt{z}}^1 \int_{\sqrt{z}}^1 x\,dxdydz.$$