Solutions final test Advanced Calculus (2DBN10)
May 2022

No rights can be derived from these solutions.

1. The characteristic polynomial is $z \mapsto z^3+5z^2+9z$. It has roots $\lambda_1 = 0$, $\lambda_{2,3} = (-5\pm\sqrt{11}i)/2$, so the (real) general solution is

$$y(t) = C_1 + e^{-5t/2}(C_2\cos(\sqrt{11}t/2) + C_3\sin(\sqrt{11}t/2)), \qquad C_{1,2,3} \in \mathbb{R}.$$

Fix $C_{1,2,3} \in \mathbb{R}$. Then for $t > 0$

$$|y(t)| \leq |C_1| + e^{-5t/2}(|C_2| + |C_3|) \leq |C_1| + |C_2| + C_3|,$$

therefore there is no solution y that satisfies $y(t) \to \infty$ as $t \to \infty$.

2. **a)** Let $F := \mathcal{L}[f]$. Then

$$F(s) = \int_0^1 e^{-st}\,dt = \frac{1-e^{-s}}{s}.$$

b) Let $Y := \mathcal{L}[y]$. Then

$$(s^2+4)Y(s) = F(s) + sy(0) + y'(0) = \frac{1-e^{-s}}{s} + s,$$

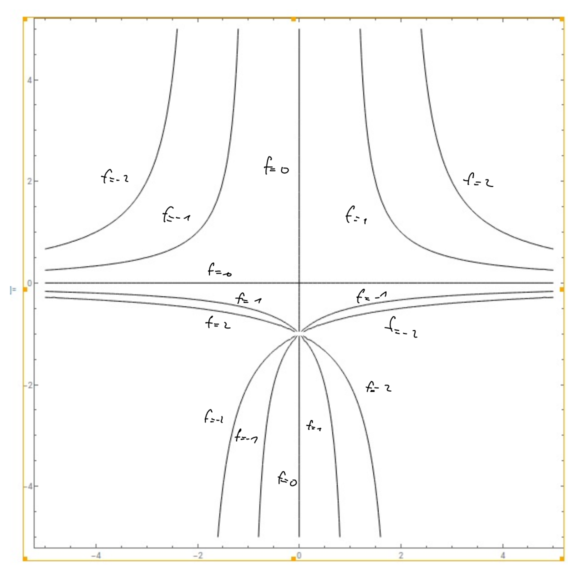
so

$$Y(s) = \frac{s^2+1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} + \frac{3s}{s^2+4}\right) + \frac{e^{-s}}{4}\left(-\frac{1}{s} + \frac{s}{s^2+4}\right)$$

and after inverse Laplace transform

$$y(s) = \begin{cases} \frac{1}{4}(1+3\cos(2t)) & (t \leq 1), \\ \frac{3}{4}\cos(2t) + \frac{1}{4}\cos(2t-2) & (t > 1). \end{cases}$$

3. **a)**



b) $z = -\frac{1}{2} + \frac{1}{2}(x+1) - \frac{1}{4}(y-1)$

c) $\frac{1}{2}(x+1) - \frac{1}{4}(y-1) = 0$

4. Using the standard expansions

$$e^y = 1 + y + \frac{y^2}{2} + O(y^3),$$
$$\frac{1}{2+z} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4}\right) + O(z^3)$$

we get

$$f(x,y) = \frac{1}{2 + x + y + \frac{y^2}{2} + O(y^3)} = \frac{1}{2}\left(1 - \frac{x}{2} - \frac{y}{2} + \frac{x^2}{4} + \frac{xy}{2}\right) + O(|(x,y)|^3).$$

The Taylor polynomial can also be found from the standard formula via partial derivatives.

5. To find the critical points we calculate

$$\nabla f(x,y) = \begin{pmatrix} 2y(x+y-1) \\ x(x+4y-2) \end{pmatrix}.$$

In the interior of D, we have $x \neq 0$ and $y \neq 0$, hence the only critical point there satisfies $x + y = 1$ and $x + 4y = 2$, i.e. $(x,y) = (2/3, 1/3)$. The corresponding function value is $f(2/3, 1/3) = -4/27$.

Moreover, $f(x,y) = xy(x + 2y - 2)$, and the boundary segments of D lie on the lines $x = 0$, $y = 0$, $x + 2y - 2 = 0$, respectively. Hence $f(x,y) = 0$ for all points (x,y) on the boundary of D.

So the global maximum is 0, taken in any boundary point, and the global minimum is $f(2/3, 1/3) = -4/27$.

6. The Lagrange equations are

$$\begin{pmatrix} y + 2z \\ x \\ 2x \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

If $\lambda = 0$ then $x = 0$ and hence $f(x,y,z) = 0$ which cannot correspond to the maximum. If $\lambda \neq 0$ then $z = 2y$ and further

$$2\lambda xy = y^2 + 2yz = 5y^2 = x^2,$$

so

$$x^2 + y^2 + z^2 = 6y^2 + 4y^2 = 1,$$

hence $y^2 = 1/10$, $x^2 = 1/2$, $z^2 = 2/5$. This yields the four solutions

$$(x,y,z) = \pm(\pm 1/\sqrt{2},\ 1/\sqrt{10},\ \sqrt{2/5}).$$

The maximum value is $\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{10}} + \sqrt{\frac{8}{5}}\right)$, taken in the two points $\pm(1/\sqrt{2},\ 1/\sqrt{10},\ \sqrt{2/5})$.

7. **a)** Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be given by

$$F(x,y,u,v) = \begin{pmatrix} x^2 + y^2 + u - v - 2 \\ x^3 + y^3 + u^2 + v^2 \end{pmatrix}.$$

Then F is continuously differentiable, $F(x_0, y_0, u_0, v_0) = 0$,

$$D_{(x,y)}F(x,y,u,v) = \begin{pmatrix} 2x & 2y \\ 3x^2 & 3y^2 \end{pmatrix},$$

and

$$D_{(x,y)}F(x_0,y_0,u_0,v_0) = \begin{pmatrix} 2 & -2 \\ 3 & 3 \end{pmatrix},$$

which is invertible. Hence the statement follows from the Implicit Function theorem.

b) From $F(\xi(u,v),\eta(u,v),u,v)=0$ we get by differentiation with respect to u and v at $(u_0,v_0)=(0,0)$ with respect to u and v

$$\begin{pmatrix} 2 & -2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \partial_u\xi & \partial_v\xi \\ \partial_u\eta & \partial_v\eta \end{pmatrix}(0,0)=\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},$$

hence

$$\begin{pmatrix} \partial_u\xi & \partial_v\xi \\ \partial_u\eta & \partial_v\eta \end{pmatrix}(0,0)=\frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

8. Cylindrical coordinates:

$$\int_0^{2\pi}\int_1^2\int_{2/r}^{3-r} r\,dz dr d\theta=\int_0^{2\pi}\int_1^2\int_{2/z}^{3-z} r\,dr dz d\theta=\frac{\pi}{3}.$$