Solutions final test Advanced Calculus (2DBN10)
November 2022

No rights can be derived from these solutions.

1. The characteristic polynomial of the coefficient matrix is $z \mapsto z^2 - 2z + 2$. Its eigenvalues are $\lambda_{1,2} = 1 \pm i$, and corresponding eigenvectors are (any nonzero complex multiples of) $v_{1,2} = (1,\ 2 \pm i)$. The real solution space is spanned by the real and imaginary part of the complex mode

$$t \mapsto e^{\lambda_1 t} v_1 = e^t \begin{pmatrix} \cos t + i \sin t \\ (2+i)(\cos t + i \sin t) \end{pmatrix} = e^t \begin{pmatrix} \cos t + i \sin t \\ 2\cos t - \sin t + i(\cos t + 2 \sin t) \end{pmatrix}.$$

So

$$y_1 = e^t \left(C_1 \begin{pmatrix} \cos t \\ 2\cos t - \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t + 2 \sin t \end{pmatrix} \right), \qquad C_{1,2} \in \mathbb{R}.$$

2. **a)** The characteristic polynomial is $z \mapsto z^2 + 3z + 2$ with roots $\lambda_1 = -2, \lambda_2 = -1$. The solution y_h to the homogenous equation is therefore given by

$$y_h(t) = C_1 e^{-2t} + C_2 e^{-t}, \qquad C_{1,2} \in \mathbb{R}.$$

To find a particular solution to the inhomogenous problem we use the ansatz

$$y_p(t) = Ate^{-t}$$

and find $A = 1$. So the general solution to the inhomogenous problem is

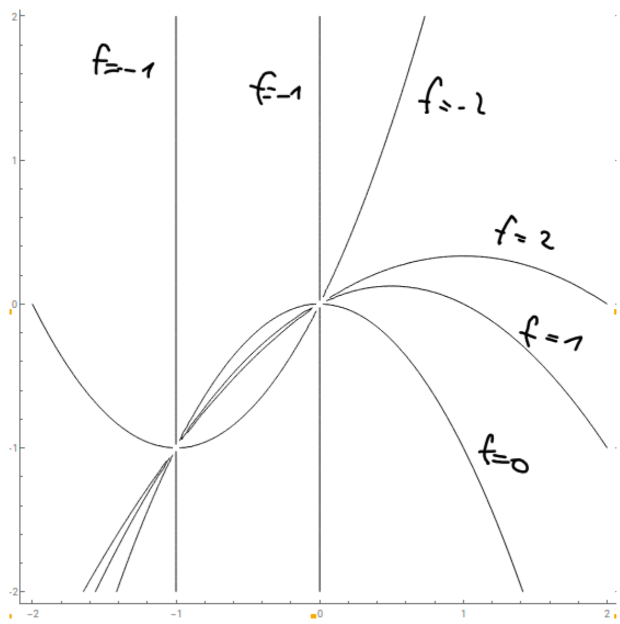
$$y(t) = C_1 e^{-2t} + (t + C_2)e^{-t}, \qquad C_{1,2} \in \mathbb{R}.$$

b) By analogous calculations as in part a), we find that there are $C_{1,2} \in \mathbb{R}$ such that for $t \geq 1$

$$u(t) = C_1 e^{-2t} + (et + C_2)e^{-t}.$$

As $e^{-t}, te^{-t}, e^{-2t} \to 0$ for $t \to +\infty$ we conclude $u(t) \to 0$ for $t \to +\infty$.

3. **a)**



b) The linearization p is given by

$$p(x,y) = f(0,-1) + \partial_x f(0,-1)x + \partial_y f(0,-1)(y+1) = -1 + x.$$

4. By the chain rule,

$$\begin{aligned}\partial_u f(u,v) &= 2u\partial_1 g(u^2-v^2, 2uv) + 2v\partial_2 g(u^2-v^2, 2uv),\\ \partial_v f(u,v) &= -2v\partial_1 g(u^2-v^2, 2uv) + 2u\partial_2 g(u^2-v^2, 2uv).\end{aligned}$$

In particular, for $u = v = \frac{1}{2}$,

$$\begin{aligned}\partial_u f(\tfrac{1}{2},\tfrac{1}{2}) &= \partial_1 g(0,\tfrac{1}{2}) + \partial_2 g(0,\tfrac{1}{2}) = 2,\\ \partial_v f(\tfrac{1}{2},\tfrac{1}{2}) &= -\partial_1 g(0,\tfrac{1}{2}) + \partial_2 g(0,\tfrac{1}{2}) = 3.\end{aligned}$$

Solving this linear system yields $\nabla g(0,\frac{1}{2}) = \frac{1}{2}(-1,5)^\top$.

5. The critical points are found as solutions of the system of equations

$$\nabla f(x,y) = \begin{pmatrix} 2xy \\ x^2+3y^2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the first equation we have $x = 0$ or $y = 0$.

1. If $x = 0$, we find from the second equation the critical points $(0, \pm\sqrt{1/3})$.

2. If $y = 0$, we find from the second equation the critical points $(\pm 1, 0)$.

The Hessian is

$$H(f)(x,y) = \begin{pmatrix} 2y & 2x \\ 2x & 6y \end{pmatrix}.$$

The second derivative test yields that $(0,\sqrt{1/3})$ is a local minimum, $(0,-\sqrt{1/3})$ is a local maximum, and $(\pm 1, 0)$ are saddle points.

6. The Lagrange equations are

$$\begin{pmatrix} x^2 \\ 2y \\ 1 \end{pmatrix} = 2\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x^2+y^2+z^2 = 2.$$

To find all solutions of these equations, we distinguish the following cases:

1. $y \neq 0$: Then $\lambda = 1$, $z = 1/2$, and $x(x-2) = 0$, hence $x = 0$ or $x = 2$.

1.1. $x = 2$: Then there are no solutions to the equation $x^2+y^2+z^2 = 2$.

1.2. $x = 0$: Then $y = \pm\sqrt{7/4}$. The points $(0, \pm\sqrt{7/4}, 1/2)$ indeed satisfy the Lagrange equations (with $\lambda = 1$), with function value $9/4$.

2. $y = 0$:

2.1. $x = 0$: Then $z = \pm\sqrt{2}$. The points $(0,0,\pm\sqrt{2})$ indeed satisfy the Lagrange equations (with $\lambda = \sqrt{2}/4$), with function value $\pm\sqrt{2}$

2.2. $x \neq 0$: Then $\lambda = x/2$ and $z = 1/(2\lambda) = 1/x$. So $x^2+z^2 = x^2+x^{-2} = 2$, which implies $x = z = \pm 1$. The points $\pm(1,0,1)$ indeed satisfy the Lagrange equations (with $\lambda = \pm 1/2$) with function value $\pm 4/3$.

So the maximal and minimal value of f taken on the sphere S are $9/4$ and $-\sqrt{2}$, respectively.

7. We have $D = \Phi(E)$ with

$$\Phi(x, y) = \begin{pmatrix} x + y \\ x - y \end{pmatrix}.$$

The mapping Φ is one-to-one, differentiable, and

$$D\Phi(x, y) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hence, by the Change-of-Variables theorem

$$\text{Area}(D) = \int\int_D 1\, dA = \int\int_{\Phi(E)} 1\, dA = \int\int_E 1 \cdot \underbrace{|\det D\Phi|}_{|-2|=2}\, dA = 2\,\text{Area}(E) = 6.$$

8. The volume is

$$\int\int\int_K dV = \int_1^2 \int_{\frac{2}{x}}^{3-x} \int_0^{3-x-y} dz dy dx = \frac{25}{6} - 6\ln 2.$$