Solutions final test Advanced Calculus (2DBN10)
January 2023

No rights can be derived from these solutions.

1. The eigenvalues of the coefficient matrix are $\lambda_{1,2} = -1 \pm \sqrt{4-2a}$.

- If $a > 2$ the eigenvalues are conjugate complex. All solutions are of the form

$$u(t) = e^{-t}(C_1 v_1 e^{i\omega t} + C_2 v_2 e^{-i\omega t})$$

with $\omega := \sqrt{2a-4}$ and suitable vectors $v_{1,2} \in \mathbb{C}^2$ and $C_{1,2} \in \mathbb{C}$. So

$$|u(t) \leq Ce^{-t} \to 0 \quad \text{as } t \to +\infty.$$

- If $3/2 < a < 2$ both eigenvalues are real and negative. All solutions are of the form

$$u(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$$

with suitable vectors $v_{1,2} \in \mathbb{C}^2$ and $C_{1,2} \in \mathbb{C}$. So

$$|u(t)| \leq Ce^{\lambda_1 t} \to 0 \quad \text{as } t \to +\infty.$$

- If $a \leq 3/2$ then $\lambda_1 \geq 0$. Let v_1 be a corresponding eigenvector. The solution u given by

$$u(t) = v_1 e^{\lambda_1 t}$$

does not approach 0 as $t \to +\infty$.

(Not demanded: For $a = 2$, all solutions converge to 0 as well.)

2. **a)** By partial fraction decomposition,

$$\frac{s^2+13s+15}{s^3+4s^2+5s} = \frac{3}{s} + \frac{-2s+1}{s^2+4s+5} = \frac{3}{s} + \frac{-2(s+2)+5}{(s+2)^2+1}.$$

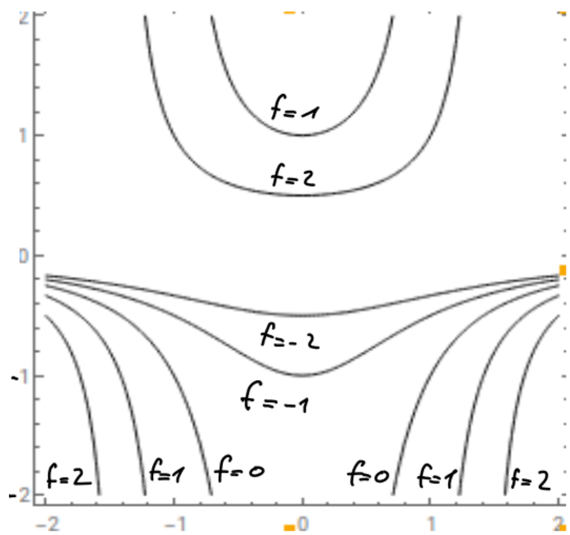
Using the table we find for the inverse transform $f = \mathcal{L}^{-1}[F]$

$$f(t) = 3 + e^{-2t}(-2\cos t + 5\sin t).$$

b) Define $H : (0,\infty) \longrightarrow \mathbb{R}$ by $H(s) = F(s)/s$ and let $h := \mathcal{L}^{-1}[H]$ be the inverse Laplace transform of H. Then $G(s) = H(s/2)/2$, and by the calculation rules

$$h(t) = \int_0^t f(\tau)\,d\tau,$$
$$g(t) = h(2t) = \int_0^{2t} f(\tau)\,d\tau.$$

3. **a)**



b) $p(x,y) = \frac{1}{2} + 2(x-1) - \frac{1}{4}(y+2)$

c) $2(x-1) - \frac{1}{4}(y+2) = 0$, or $8x - y = 10$.

4. Standard expansions up to quadratic terms in $(x-1)$ and $(y-1)$:

$$
\begin{aligned}
\frac{1}{y} &= \frac{1}{1+(y-1)} = 1 - (y-1) + (y-1)^2 + \dots, \\
\frac{x}{y} &= (1+(x-1))\frac{1}{y} = 1 + (x-1) - (y-1) - (x-1)(y-1) + (y-1)^2 + \dots, \\
e^{\frac{x}{y}} &= e \cdot e^{\frac{x}{y}-1} \\
&= e\big[1 + (x-1) - (y-1) - (x-1)(y-1) + (y-1)^2 + \dots + \tfrac{1}{2}\big((x-1) - (y-1) + \dots\big)^2 + \dots\big] \\
&= e\big[1 + (x-1) - (y-1) + \tfrac{1}{2}(x-1)^2 - 2(x-1)(y-1) + \tfrac{3}{2}(y-1)^2\big] + \dots,
\end{aligned}
$$

hence the Taylor polynomial is given by

$$T_{2,(1,1)}(x,y) = e\big[1 + (x-1) - (y-1) + \tfrac{1}{2}(x-1)^2 - 2(x-1)(y-1) + \tfrac{3}{2}(y-1)^2\big].$$

This result can alternatively be obtained by calculating the partial derivatives at $(1,1)$ and using the standard formula.

5. 1. Critical points of f satisfy

$$\nabla f(x,y) = \begin{pmatrix} 2x(1+y) \\ 2y + x^2 \end{pmatrix} = 0,$$

hence $x = 0$ or $y = -1$.

If $x = 0$ then $y = 0$. The point $(0,0)$ is a critical point in D with $f(0,0) = 0$.

If $y = -1$ then $x = \pm\sqrt{2}$. The points $(\pm\sqrt{2}, -1)$ are critical points in D with $f(\pm\sqrt{2}, -1) = 1$.

2. We parameterize the boundary of D by $x = 2\cos\theta$, $y = 2\sin\theta$, $\theta \in [0, 2\pi)$. For points on this boundary,

$$f(x,y) =: g(\theta) = 4 + 8\cos^2\theta \sin\theta.$$

We find extrema of g by solving

$$g'(\theta) = 8\cos\theta(-2\sin^2\theta + \cos^2\theta) = 0.$$

This equation implies $\cos\theta = 0$ or $\cos^2\theta = 2\sin^2\theta$.

If $\cos\theta = 0$ then $x = 0$, hence $y = \pm 2$, with function values $f(0, \pm 2) = 4$.

If $\cos^2\theta = 2\sin^2\theta$ then $x^2 = 2y^2$, hence $3y^2 = 4$, i.e. $y = \pm\sqrt{4/3}$, $x = \pm\sqrt{8/3}$, with function values $4 \pm \frac{16}{9}\sqrt{3}$.

(Alternatively, the extrema of f on the boundary of D can also be obtained by the method of Lagrange multipliers.)

3. Comparing the function values in the candidate points yields that the global minimum of f on D is 0, taken at $(0, 0)$, and the global maximum is $4 + \frac{16}{9}\sqrt{3}$, taken at $(\pm\sqrt{8/3}, \sqrt{4/3})$.

6. We have to minimize and maximize $f(x, y, z) = z$ under the restrictions

$$\begin{aligned} g_1(x, y, z) &= x + y + z - 1 = 0, \\ g_2(x, y, z) &= x^2 + y^2 - z = 0. \end{aligned}$$

The Lagrange equations for this problem are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix}.$$

As the three vectors in this system are linearly dependent, we have that

$$\det \begin{pmatrix} 0 & 1 & 2x \\ 0 & 1 & 2y \\ 1 & 1 & -1 \end{pmatrix} = 2y - 2x = 0,$$

i.e. $x = y$. This yields $2x + z = 1$ and $z = 2x^2$ from the restrictions, hence $2x^2 + 2x = 1$ with the solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{3}}{2},$$

corresponding to the minimum and maximum value

$$z_{1,2} = 1 - 2x_{1,2} = 2 \mp \sqrt{3}.$$

7. **a)** To apply the Implicit Function theorem (IFT) with $t_0 = 0$ we need (x_0, y_0) such that

$$\begin{aligned} \sin x_0 + 2\sin y_0 &= 0, \\ e^{x_0} + e^{y_0} &= 2, \end{aligned}$$

so we can choose $(x_0, y_0) = (0, 0)$.
Define $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$F(x, y, z) = \begin{pmatrix} \sin x + 2\sin y - t \\ e^x + e^y - t^2 - 2 \end{pmatrix}.$$

Then $F(x_0, y_0, t_0) = F(0, 0, 0) = 0$, and the Jacobian

$$D_{(x,y)}F(x, y, t)|_{(0,0,0)} = \begin{pmatrix} \cos x & 2\cos y \\ e^x & e^y \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

is regular. The existence of the functions ξ and η with the announced properties follows from applying the IFT.

b) Differentiating $F(\xi(t), \eta(t), t) = 0$ with respect to t at $t = 0$ and using that $\xi(0) = x_0 = 0$, $\eta(0) = y_0 = 0$ yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi'(0) \\ \eta'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with solution $\xi'(0) = -1$, $\eta'(0) = 1$.

8. Cylindrical coordinates:

$$\iiint_K z\,dV = \int_0^{2\pi} \int_0^1 r \int_{\sqrt{r}}^1 z\,dzdrd\theta \quad \text{or} \quad \iiint_K z\,dV = \int_0^{2\pi} \int_0^1 z \int_0^{z^2} r\,drdzd\theta = \pi/6.$$