Solutions final test Advanced Calculus (2DBN10)
November 2023

No rights can be derived from these solutions.

1. **a)** The eigenvalues of the coefficient matrix are $\lambda_1 = -1$, $\lambda_2 = 3$ with corresponding eigenvectors $(1,1)^\top$ and $(3,2)^\top$. So all (real) solution are given by

$$y(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}, \qquad C_{1,2} \in \mathbb{R}.$$

These solutions go to 0 for $t \to \infty$ if and only if $C_2 = 0$. In this case we have

$$y(0) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

i.e. $z_1 = z_2$.

b) Ansatz via variation of parameters:

$$y_p(t) = C_1(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2(t) \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}.$$

This leads to the linear system

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t}\dot{C}_1(t) \\ e^{3t}\dot{C}_2(t) \end{pmatrix} = \begin{pmatrix} e^t \\ 1 \end{pmatrix}$$

with solution

$$\begin{aligned} \dot{C}_1(t) &= -2e^{2t} + 3e^t, \\ \dot{C}_2(t) &= e^{-2t} - e^{-3t} \end{aligned}$$

and after integration

$$\begin{aligned} C_1(t) &= -e^{2t} + 3e^t, \\ C_2(t) &= -\tfrac{1}{2}e^{-2t} + \tfrac{1}{3}e^{-3t} \end{aligned}$$

(with integration constants set to 0 as we seek a particular solution only). So

$$y_p(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-e^t + 3) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} \left(-\tfrac{1}{2}e^t + \tfrac{1}{3} \right) = \begin{pmatrix} -\frac{5}{2}e^t + 4 \\ -2e^t + \frac{11}{3}. \end{pmatrix}$$

2. **a)** Rewrite

$$F(s) = \underbrace{\frac{1}{(s+1)^2+2}}_{F_1(s)} - \underbrace{\frac{e^{-s/2}}{(s+1)^2+2}}_{F_2(s)}.$$

The calculation rules yield

$$f_1(t) := \mathcal{L}^{-1}[F_1](t) = \frac{e^{-t}}{\sqrt{2}} \sin(\sqrt{2}t),$$

$$f_2(t) := \mathcal{L}^{-1}[F_2](t) = \begin{cases} 0 & \text{if } t < 1/2, \\ \frac{e^{-(t-1/2)}}{\sqrt{2}} \sin(\sqrt{2}(t-1/2)) & \text{if } t \geq 1/2, \end{cases}$$

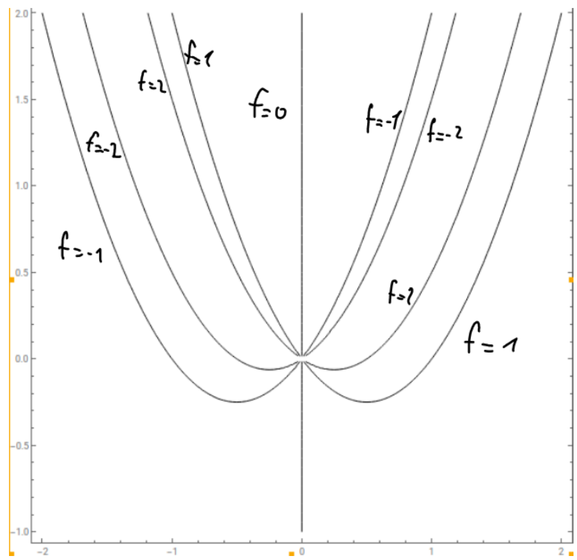
so

$$\mathcal{L}^{-1}[F_2](t) = f_1(t) - f_2(t) = \begin{cases} \frac{e^{-t}}{\sqrt{2}} \sin(\sqrt{2}t) & \text{if } t < 1/2, \\ \frac{e^{-t}}{\sqrt{2}} \left(\sin(\sqrt{2}t) - e^{1/2} \sin(\sqrt{2}(t-1/2)) \right) & \text{if } t \geq 1/2. \end{cases}$$

b) As f' is a bounded continuous function, its Laplace transform converges to 0 as $s \to \infty$. Hence

$$sF(s) = \mathcal{L}[f'](s) + f(0) \overset{s\to\infty}{\longrightarrow} f(0).$$

3. **a)**



b) $z = \frac{2}{3} - \frac{5}{9}(x-2) + \frac{2}{9}(y-1)$

c) $-\frac{5}{9}(x-2) + \frac{2}{9}(y-1) = 0$, or $5x - 2y = 8$.

4. Let $z = x\cos y - 1$. Then, by standard expansions,

$$\begin{aligned} z &= (1+(x-1))(1-y^2/2+O(y^4)) - 1 = (x-1) - y^2/2 + O(|(x-1,y)|^3), \\ z^2 &= (x-1)^2 + O(|(x-1,y)|^3), \\ z^3 &= O(|(x-1,y)|^3), \\ f(x,y) &= 1/(1-z) = 1 + z + z^2 + O(z^3) = 1 + (x-1) - y^2/2 + (x-1)^2 \\ &\quad +O(|(x-1,y)|^3), \end{aligned}$$

and the Taylor polynomial is given by

$$T(x,y) = 1 + (x-1) - y^2/2 + (x-1)^2.$$

This result can alternatively be obtained by calculating the partial derivatives at $(1,1)$ and using the standard formula.

5. **a)** Critical points of f satisfy

$$\nabla f(x,y) = \begin{pmatrix} 4x^3 - 8y \\ 4y^3 - 8x \end{pmatrix} = 0,$$

hence $x^3 = 2y$ and $y^3 = 2x$. So $x^9 = 8y^3 = 16x$, and hence either $x = 0$ or $x^8 = 16$, i.e. $x = \pm\sqrt{2}$.

If $x = 0$ then $y = 0$, and indeed $\nabla f(0,0) = 0$. If $x = \pm\sqrt{2}$ then $y = \pm\sqrt{2}$, and indeed $\nabla f(\pm(\sqrt{2},\sqrt{2})) = 0$. For the Hessian we get

$$D^2 f(x,y) = \begin{pmatrix} 12x^2 & -8 \\ -8 & 12y^2 \end{pmatrix}, \quad D^2 f(0,0) = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix}, \quad D^2 f(\pm(\sqrt{2},\sqrt{2})) = \begin{pmatrix} 24 & -8 \\ -8 & 24 \end{pmatrix},$$

and the second derivative test yields that $(0,0)$ is a saddle point and $\pm(\sqrt{2},\sqrt{2})$ are places of (strict) local minima.

b) As e.g. $f(x,0) = x^4$ gets arbitrarily large for x large, f has no upper bounds and therefore no global maximum.

Alternative: As $\mathbb{R}^2$ has no boundary points, a global (hence also local) maximum would be taken at a critical point. However, we have already seen that there is no critical point at which the function takes a local maximum.

6. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$g(x, y) = (x + 2)^2 + y^2 - 4.$$

The Lagrange equations are

$$\nabla f(x, y) = e^x \begin{pmatrix} x^2 - y^2 + 2x \\ -2y \end{pmatrix} = \lambda \nabla g(x, y) = \lambda \begin{pmatrix} 2(x+2) \\ 2y \end{pmatrix}, \qquad g(x, y) = 0.$$

We distinguish the cases $y = 0$ and $y \neq 0$.

1. If $y = 0$ then $x = 0$ or $x = -4$ because of $g(x, 0) = 0$.

1.1. If $x = 0$ then $\lambda = 0$, and $(x, y, \lambda) = (0, 0, 0)$ indeed solves the Lagrange equations. The corresponding function value is $f(0, 0) = 0$.

1.2. If $x = -4$ then $\lambda = -2e^{-4}$, and $(x, y, \lambda) = (-4, 0, -2e^{-4})$ indeed solves the Lagrange equations. The corresponding function value is $f(-4, 0) = 16e^{-4}$.

2. If $y \neq 0$ then $\lambda = -e^x \neq 0$, and hence $x^2 - y^2 + 2x = -2(x + 2)$, i.e. $(x + 2)^2 = y^2$. Together with $g(x, y) = 0$ this implies $(x + 2)^2 = y^2 = 2$. This yields the four solutions

$$\begin{aligned} (x, y, \lambda) &= (-2 - \sqrt{2}, \pm\sqrt{2}, -e^{-(2-\sqrt{2})}), \\ (x, y, \lambda) &= (-2 + \sqrt{2}, \pm\sqrt{2}, -e^{-(2+\sqrt{2})}), \end{aligned}$$

with corresponding function values

$$\begin{aligned} f(-2 - \sqrt{2}, \pm\sqrt{2}) &= 4e^{-2-\sqrt{2}}(1 + \sqrt{2}), \\ f(-2 + \sqrt{2}, \pm\sqrt{2}) &= 4e^{-2+\sqrt{2}}(1 - \sqrt{2}). \end{aligned}$$

Comparison of the function values yields that the global maximum is $4e^{-2-\sqrt{2}}(1 + \sqrt{2})$ and the global minimum is $4e^{-2+\sqrt{2}}(1 - \sqrt{2})$.

7. Using the change of variables theorem, we get

$$\text{Area}(\Phi(D)) = \iint_{\Phi(D)} 1\, dA = \iint_D |\det D\Phi|\, dA = 2 \iint_D (x^2+y^2)\, dA = 2 \int_0^1 \int_0^{1-x} (x^2+y^2)\, dA = 1/3.$$

8. "adapted" cylindrical coordinates $((x, y, z) = (x, r\cos\theta, r\sin\theta))$:

$$\iiint_K x\, dV = \int_0^{2\pi} \int_1^2 x \int_0^{\sqrt{4-x^2}} r\, dr dx d\theta \quad \text{or} \quad \iiint_K x\, dV = \int_0^{2\pi} \int_0^{\sqrt{3}} r \int_1^{\sqrt{4-r^2}} x\, dx dr d\theta = 9\pi/4.$$