Solutions final test Advanced Calculus (2DBN10)
January 2024

No rights can be derived from these solutions.

1. The eigenvalues of the coefficient matrix are $\lambda_{1,2} = 2 \pm i$. An eigenvector corresponding to $\lambda_1 = 2 + i$ is $(2, 3 + i)$. So the general real solution is

$$\begin{aligned} y(t) &= C_1 \operatorname{Re}\left[\begin{pmatrix} 2 \\ 3+i \end{pmatrix} e^{(2+i)t}\right] + C_2 \operatorname{Im}\left[\begin{pmatrix} 2 \\ 3+i \end{pmatrix} e^{(2+i)t}\right] \\ &= e^{2t}\left[C_1 \begin{pmatrix} 2\cos t \\ 3\cos t - \sin t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin t \\ \cos t + 3\sin t \end{pmatrix}\right]. \end{aligned}$$

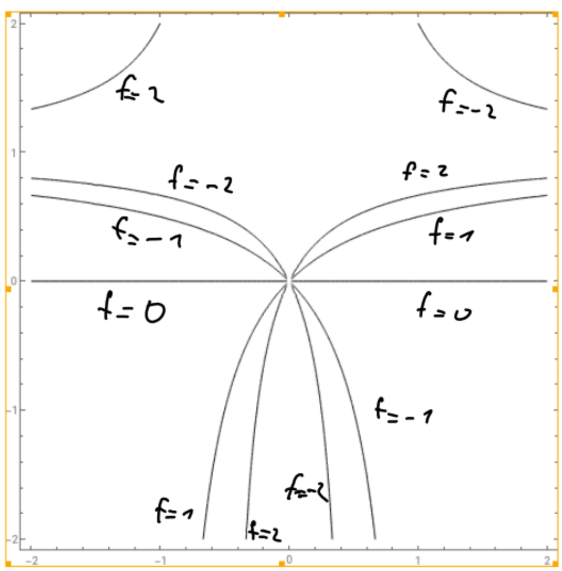
These solutions satisfy $y(t) \to 0$ for $t \to -\infty$ and are therefore not periodic (unless $y \equiv 0$.)

2. The characteristic polynomial has the simple root $\lambda_1 = 1$ and the double root $\lambda_{2,3} = -2$. Accordingly, a suitable ansatz is

$$y_p(t) = At^2 e^{-2t} + B.$$

Direct calculation yields $A = -1/6$, $B = -1/4$.

3. **a)**



b) $z = -2 + 2(x-1) + y - 2$

c) $2(x-1) + y - 2 = 0$, or $2x + y = 4$.

4. Expressing $D_v f(0,0)$ and $D_w f(0,0)$ in terms of the partial derivatives $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ yields

$$\begin{aligned} \partial_1 f(0,0) + \partial_2 f(0,0) &= 3, \\ 2\partial_1 f(0,0) - \partial_2 f(0,0) &= -3. \end{aligned}$$

Solving this system yields $\nabla f(0,0) = (\partial_1 f(0,0), \partial_2 f(0,0))^\top = (0,3)^\top$.

5. Critical points of f satisfy

$$\nabla f(x,y) = \begin{pmatrix} 1 - 2xy \\ 1 - x^2 \end{pmatrix} = 0,$$

hence $(x, y) = \pm(1, 1/2)$. Only the point $(1, 1/2)$ lies in D, with $f(1, 1/2) = 1$.

We now consider the boundary line segments:

1. Let

$$g_1(x) := f(x, 0) = x, \quad 0 \le x \le 2.$$

The maximal value of f on the line segment from $(0, 0)$ to $(2, 0)$ is $f(2, 0) = 2$.

The minimal value of f on the line segment from $(0, 0)$ to $(2, 0)$ is $f(0, 0) = 0$.

2. Let

$$g_2(y) := f(0, y) = y, \quad 0 \le y \le 2.$$

The maximal value of f on the line segment from $(0, 0)$ to $(0, 2)$ is $f(0, 2) = 2$.

The minimal value of f on the line segment from $(0, 0)$ to $(0, 2)$ is $f(0, 0) = 0$.

3. Let

$$g_3(x) := f(x, 2 - x) = 2 - x^2(2 - x) = x^3 - 2x^2 + 2, \quad 0 \le x \le 2.$$

Then $g_3'(x) = x(3x - 4)$ and $g_3'(x) = 0$ only if $x = 0$ or $x = 4/3$.

The maximal value of f on the line segment from $(0, 2)$ to $(2, 0)$ is $f(0, 2) = f(2, 0) = 2$.

The minimal value of f on the line segment from $(0, 2)$ to $(2, 0)$ is $f(4/3, 2/3) = 22/27$.

So the maximal value of f on D is $f(0, 2) = f(2, 0) = 2$ and the minimal value of f on D is $f(0, 0) = 0$.

6. The Lagrange equations are

$$\begin{pmatrix} y \\ x - 1 \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}, \quad x^2 + y^2 + z^2 = 1$$

Because of $z = \lambda z$ we either have $z = 0$ or $\lambda = 1$.

1. If $z = 0$ then $1 - x^2 = y^2 = 2\lambda xy = x^2 - x$, i.e. $2x^2 - x - 1 = 0$ with solutions $x_1 = -1/2$, $x_2 = 1$.

1.1. If $x = 1/2$ then $y = \lambda = \pm\sqrt{3}/2$. This yields two solutions to the Lagrange equations with $f(1/2, \pm\sqrt{3}/2, 0) = \mp 3\sqrt{3}/4$.

1.2. If $x = 1$ then $y = \lambda = 0$. This yields a solution to the Lagrange equations with $f(1, 0, 0) = 0$.

2. If $\lambda = 1$ then $4x = 2y = x - 1$, i.e. $x = -1/3$, $y = -2/3$, $z = \pm 2/3$. This yields two solutions to the Lagrange equations with $f(-1/3, -2/3, \pm 2/3) = 4/3$.

Comparing the values in the solution points, we find that the maximal value of f is $4/3$ and the minimal value is $-3\sqrt{3}/4$.

7. a) With $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$F(t, x, y) = \begin{pmatrix} e^{xt} + x + y - 3 \\ e^{yt} + x^2 + y^3 - 3 \end{pmatrix}$$

the conditions are:
$F(t_0, x_0, y_0) = 0$, i.e.

$$\begin{aligned} e^{x_0 t_0} + x_0 + y_0 &= 3, \\ e^{y_0 t_0} + x_0^2 + y_0^3 &= 3, \end{aligned}$$

and

$$DF_{(x,y)}(t_0, x_0, y_0) = \begin{pmatrix} t_0 e^{x_0 t_0} + 1 & 1 \\ 2x_0 & t_0 e^{y_0 t_0} + 3y_0^2 \end{pmatrix}$$

invertible. Under these assumptions, the Implicit Function theorem ensures the unique solvability of the given system near (t_0, x_0, y_0) for $x = \xi(t)$, $y = \eta(t)$.

b) We indeed have $F(0,1,1) = 0$ and

$$D_{(x,y)}F(0,1,1) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

which is invertible (e.g. because its determinant is 1).
From the chain rule we get

$$D_{(x,y)}F(0,1,1) \begin{pmatrix} \xi'(0) \\ \eta'(0) \end{pmatrix} = -D_t F(0,1,1) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solving this system yields $\xi'(0) = -2$, $\eta'(0) = 1$.
So the linearizations around $t = 0$ are

$$\xi(t) \approx 1 - 2t, \quad \eta(t) \approx 1 + t.$$

8. Cylindrical coordinates:

$$\mathrm{Vol}(K) = \iiint_K dV = \int_0^{\pi/4} \int_0^1 \int_{z(1-z)}^{2z(1-z)} r\, dr dz d\theta = \pi/80.$$

9. "adapted" cylindrical coordinates ($(x, y, z) = (x, r\cos\theta, r\sin\theta)$):

$$\iiint_K x\, dV = \int_0^{2\pi} \int_1^2 x \int_0^{\sqrt{4-x^2}} r\, dr dx d\theta \quad \text{or} \quad \iiint_K x\, dV = \int_0^{2\pi} \int_0^{\sqrt{3}} r \int_1^{\sqrt{4-r^2}} x\, dx dr d\theta = 9\pi/4.$$