ANALYSIS PROOFS - BEST PRACTICES

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These notes are meant as a guide to proving mathematical statements in analysis. At the end of the notes, you find a list of **best practices**. It is my experience that if you follow these best practices, proving statements becomes easier, and your proofs are clearer and easier to read.

A typical statement in analysis is that the sequence

$$\left(\frac{1}{n} : \ n \in \mathbb{N}\right)$$

converges to $0 \in \mathbb{R}$ as $n \to \infty$. We write this as

$$\forall \epsilon > 0 : \ \exists n_0 \in \mathbb{N} : \ \forall n \geq n_0 : \ |1/n - 0| < \epsilon.$$

I believe that one of the main difficulties of Analysis 1 and 2 is learning to deal with a large number of quantifiers (that is, larger than 1).

First of all, it helps me to put some extra brackets:

$$\forall \epsilon > 0 : \ \{\exists n_0 \in \mathbb{N} : \ \{\forall n \geq n_0 : \ \{|1/n - 0| < \epsilon\}\}\}$$

Everything that is contained between a pair of brackets, is a mathematical statement in itself.

To further clarify the nested structure of one mathematical statement inside the other, I like to use indentation, and I prefer to use words over quantifier-symbols

for all $\epsilon > 0$:
  there exists an $n_0 \in \mathbb{N}$ such that :
    for all $n \geq n_0$:
      $|1/n - 0| < \epsilon$

Alternatively we can use both brackets and indentation[1].

for all $\epsilon > 0$
{
  there exists an $n_0 \in \mathbb{N}$ such that
  {
    for all $n \geq n_0$
    {
      $|1/n - 0| < \epsilon$
    }
  }
}

Such a statement looks complicated, and you may not know how to start a proof or how to continue.

But one of the main messages of this note is that the statement itself tells you how to start, and how to continue. In fact, the statement gives you a template, where there are only a few things left for you to fill in.

The huge trick

The trick to proving a mathematical statement involving many quantifiers is to prove it *block by block*. I will explain how to directly prove "for all" statements and "there exists" statements (that is, without using a contradiction argument). As a running example, we will prove

[1]It is not a coincidence that these look like Python, C++ or Java code. However, one shouldn't think of these statements as programs, but rather as types (such as integer, double, a function type that assigns integers to integers). You can then think of the proofs of these statements as computer programs.

that

for all $\epsilon > 0$
{
  there exists an $n_0 \in \mathbb{N}$ such that
  {
    for all $n \geq n_0$
    {
      $|1/n - 0| < \epsilon$
    }
  }
}

by peeling layer after layer of the statement.

Directly proving "for all" statements

The general question is how to prove a statement of the form

$$\forall a \in A: \ \{\dots\}$$

that is how to prove

for all $a \in A: \ \{\dots\}$

In our example, we see this twice: in

for all $\epsilon > 0: \ \{\dots\}$

and in

for all $n \geq n_0: \ \{\dots\}$

By *definition*, you prove the statement

for all $a \in A: \ \{\dots\}$

in the following way. You first introduce (i.e. define) the variable $a \in A$ by writing something like

Let $a \in A$.

Next, you continue to prove whatever is inside of the block $\{\dots\}$.

For the statement

for all $\epsilon > 0 \ \{\dots\}$

this comes down to writing

Let $\epsilon > 0$.

Directly proving "there exists" statements

How to prove the statement

$\exists b \in B: \{\dots\}$

In our example, we see

there exists $n_0 \in \mathbb{N}$ such that $\{\dots\}$

The standard approach is to make a particular choice for n_0 now, and then prove whatever is inside the block $\{\dots\}$ with this choice of n_0. We would for instance write:

Choose $n_0 = 10$.

and then continue with the proof of the block $\{\dots\}$, with now n_0 fixed as 10. Making choices is hard. With a bad choice, you won't be able to prove whatever is inside the block $\{\dots\}$. In many of the proofs that you will write, this is probably the step that requires the most thinking, the most creativity.

Trying to finish the proof

Where are we now? Our proof so far consists of

Let $\epsilon > 0$.
  Choose $n_0 = 10$.

and now we need to prove the statement

for all $n \geq n_0$
{
  $|1/n - 0| < \epsilon$
}

with the only knowledge about ϵ that it is a (real) number larger than 0, and that $n_0 = 10$.

Let us stick to the recipe. We need to show a statement of the form

for all $n \geq n_0$

so we define n by writing

Let $n \geq n_0$

and continue proving the block $\{|1/n - 0| < \epsilon\}$.

Of course, now we are in big trouble. Because we chose $n_0 = 10$, we can merely guarantee that

$|1/n - 0| = 1/n \leq 1/n_0 = 1/10.$

But we do not know whether ϵ is larger than $1/10$ or not! All we know is that it is a positive real number.

We cannot prove that

$$|1/n - 0| < \epsilon$$

because we made a bad choice for n_0. This happens. It is almost impossible to figure out the proof 'linearly', in a prearranged order of steps. To know how to choose n_0, you need to know your endgame, you already need to know how you will finish the proof. This means, you need to do a lot of scratchwork.

Let us do some of that scratchwork. We can figure out what choice of n_0 *would* lead to a proof. For instance if we choose instead

$$n_0 = \lceil 1/\epsilon \rceil + 1$$

then n_0 is a natural number strictly larger than ϵ (here $\lceil 1/\epsilon \rceil$ is $1/\epsilon$ rounded up to a natural number). In that case

$$|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon.$$

It works!

But you need to present the proof following the above steps. You just make better choices.

Let's try again

We need to show

for all $\epsilon > 0$
{
  there exists an $n_0 \in \mathbb{N}$ such that
  {
    for all $n \geq n_0$
    {
      $|1/n - 0| < \epsilon$
    }
  }
}

so we write

Let $\epsilon > 0$.

and continue with the proof of the statement

there exists an $n_0 \in \mathbb{N}$ such that
{
  for all $n \geq n_0$
  {
    $|1/n - 0| < \epsilon$
  }
}

We are wiser now, and write

Choose $n_0 = \lceil 1/\epsilon \rceil + 1$

and continue with the proof of

for all $n \geq n_0$
{
  $|1/n - 0| < \epsilon$
}

Next, we write

Let $n \geq n_0$.

and we are ready to, once again, try to prove that

$|1/n - 0| < \epsilon.$

Indeed, now it is time to insert our calculation

$|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon.$

This finishes the proof. If we write everything in one go it works as follows.

Let $\epsilon > 0$.
  Choose $n_0 = \lceil 1/\epsilon \rceil + 1$.
    Let $n \geq n_0$.
      Then $|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon$.

The statement we proved already suggested to us the template

Let $\epsilon > 0$.
  Choose $n_0 = \dots$
    Let $n \geq n_0$.
      Then **show desired estimate**.

We filled it in by choosing n_0 appropriately, and making a correct estimate.

Negations and quantifiers

We write the negation of a mathematical statement as $\neg\{\dots\}$. De Morgan's laws specify how quantifiers behave under negation.

The statement

$$\neg\{\forall a \in A: \ \{\dots\}\}$$

is equivalent to

$$\exists a \in A: \{\neg\{\dots\}\}$$

Similarly, the statement

$$\neg\{\exists a \in A: \ \{\dots\}\}$$

is equivalent to

$$\forall a \in A: \ \{\neg\{\dots\}\}$$

Proofs by contradiction

When you are stuck proving something directly, it is a good idea to try to give a proof by contradiction: You assume that whatever you want to show is not true, and derive a contradiction from there. Some statements in analysis are (almost?) impossible to show without using a contradiction argument somewhere.

Best practices

(1) Write down what is **given** and what you need **to show**.

(2) To prove a statement

for all $a \in A: \ \{\dots\}$

you first introduce $a \in A$ by writing

Let $a \in A$.

and then you continue to prove $\{\dots\}$

(3) To prove a statement

there exists $a \in A$ such that : $\{\dots\}$

you make a choice for a and write

Choose $a = \dots$

and then you continue to prove $\{\dots\}$

(4) Use words such as if ... then ... in your proof rather than implication symbols $\Rightarrow$ and $\Leftrightarrow$.

(5) If the statement that you need to show is an "if and only if" statement, show the "if" and "only if" statements separately.

(6) Make sure that every variable that you are using is defined. Here there is a difference between the quantifiers "for all" and "there exists":

- After writing a sentence:

$\forall \epsilon > 0 \quad \dots$

the variable ϵ is not defined, and you cannot refer to it

- After the sentence

$\exists N \in \mathbb{N} \quad \dots$

the variable N is defined, and you can refer to it.

(7) Use your words. For example, use words to indicate whether a statement that you write down is a statement you want to show, or whether it is a statement that you assume, or whether it is a consequence of your earlier calculations.

(8) Care about your presentation of the proof. Write not only legibly, but neatly.