Proposition 1 *(Uniform convergence preserves limits)*
Let $D \subset \mathbb{R}$ and let a be an accumulation point of D, (f_n) a sequence of functions on D, $f_n \to f^*$ uniformly. Suppose $\lim_{x\to a} f_n(x) = L_n$. Then

$$\lim_{n\to\infty} L_n = L \quad \Leftrightarrow \quad \lim_{x\to a} f^*(x) = L.$$

If the limits exist, then

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x).$$

Proof: "$\Rightarrow$" Let $\varepsilon > 0$ be given. Then

$$\exists n_0 \in \mathbb{N}: \quad \forall n \geq n_0: \quad |L_n - L| < \varepsilon/3 \;\wedge\; \|f_n - f^*\|_\infty < \varepsilon/3.$$

Fix such an n_0. Then

$$\exists \delta > 0: \quad \forall x \in D: \quad 0 < |x - a| < \delta \;\Rightarrow |f_{n_0}(x) - L_{n_0}| < \varepsilon/3.$$

Fix such a δ and let $x \in D$ with $0 < |x - a| < \delta$. Then

$$|f^*(x) - L| \overset{\triangle}{\leq} \underbrace{|f^*(x) - f_{n_0}(x)|}_{\leq \|f_n - f^*\|_\infty < \varepsilon/3} + \underbrace{|f_{n_0}(x) - L_{n_0}|}_{<\varepsilon/3} + \underbrace{|L_{n_0} - L|}_{<\varepsilon/3} < \varepsilon.$$

"$\Leftarrow$" Let $\varepsilon > 0$ be given. Then

$$\exists n_0 \in \mathbb{N}: \quad \forall n \geq n_0: \quad \|f_n - f^*\|_\infty < \varepsilon/3.$$

Fix such an n_0 and let $n \geq n_0$. Because of $\lim_{x\to a} f_n(x) = L_n$ and $\lim_{x\to a} f^*(x) = L$ we have

$$\exists \delta = \delta(n) > 0: \quad \forall x \in D: \quad 0 < |x-a| < \delta \;\Rightarrow \big(|f_n(x) - L_n| < \varepsilon/3 \wedge |f^*(x) - L| < \varepsilon/3\big).$$

Now choose an $x \in D$ such that $0 < |x - a| < \delta$. Then

$$|L_n - L| \overset{\triangle}{\leq} \underbrace{|L_n - f_n(x)|}_{<\varepsilon/3} + \underbrace{|f_n(x) - f^*(x)|}_{\leq \|f_n - f^*\|_\infty < \varepsilon/3} + \underbrace{|f^*(x) - L|}_{<\varepsilon/3} < \varepsilon.$$

■