Let $I \subset \mathbb{R}$ be an interval.

Proposition 1 *Let (f_n) be a sequence of functions on I. Suppose:*

(i) *f_n continuously differentiable on I for all n*

(ii) *(f_n) pointwise convergent on I to $f : I \longrightarrow \mathbb{R}$.*

(iii) *f_n' uniformly convergent to $g : I \longrightarrow \mathbb{R}$.*

Then f is continuously differentiable on I and $f' = g$.

Proof: Because of (i), (iii), and Theorem [K] 8.3.1, g is continuous on I. Let $a \in I$ and $\varepsilon > 0$ be given. Because of the continuity of g,

$$\exists \delta > 0 : \quad \forall x \in I : \quad |x - a| < \delta \ \Rightarrow |g(x) - g(a)| < \varepsilon/4.$$

Choose such a δ and let $h \in (-\delta, \delta) \setminus \{0\}$ with $a + h \in I$. We are going to show:

$$\left| \frac{f(a+h) - f(a)}{h} - g(a) \right| < \varepsilon. \tag{1}$$

Because of (ii) and (iii) we have

$$\exists n_1 : \quad \forall n \geq n_1 : \quad \|f_n' - g\|_\infty < \varepsilon/4,$$
$$\exists n_2 : \quad \forall n \geq n_2 : \quad |f_n(a) - f(a)| < \varepsilon|h|/4,$$
$$\exists n_3 : \quad \forall n \geq n_3 : \quad |f_n(a+h) - f(a+h)| < \varepsilon|h|/4.$$

Choose $n_{1,2,3}$ correspondingly, and let $n \geq n_0 := \max\{n_1, n_2, n_3\}$. According to the Mean Value theorem, there is a c between a and $a + h$ such that

$$\frac{f_n(a+h) - f_n(a)}{h} = f_n'(c),$$

and because of $|a - c| < |h| < \delta$ we have

$$|g(c) - g(a)| < \varepsilon/4.$$

So

$$\begin{aligned}
& \left| \frac{f(a+h) - f(a)}{h} - g(a) \right| \\
\overset{\triangle}{\leq} \quad & \frac{1}{|h|}|f(a+h) - f_n(a+h)| + \left| \frac{f_n(a+h) - f_n(a)}{h} - g(a) \right| + \frac{1}{|h|}|f_n(a) - f(a)| \\
\overset{\text{MVT}}{\leq} \quad & \frac{1}{|h|}|f(a+h) - f_n(a+h)| + |f_n'(c) - g(a)| + \frac{1}{|h|}|f_n(a) - f(a)| \\
\overset{\triangle}{\leq} \quad & \underbrace{\frac{1}{|h|}|f(a+h) - f_n(a+h)|}_{<\varepsilon/4} + \underbrace{|f_n'(c) - g(c)|}_{\leq \|f_n' - g\|_\infty < \varepsilon/4} + \underbrace{|g(c) - g(a)|}_{<\varepsilon/4} \\
& + \underbrace{\frac{1}{|h|}|f_n(a) - f(a)|}_{<\varepsilon/4} < \varepsilon,
\end{aligned}$$

■

Proposition 2 *([K] 8.2.8, Cauchy's criterion)*
Let $D \subset \mathbb{R}$ and let (f_n) be a sequence of functions on D such that

$$\forall \varepsilon > 0 : \quad \exists n_0 : \quad \forall n, m \geq n_0 : \quad \|f_n - f_m\|_\infty < \varepsilon.$$

Then (f_n) is uniformly convergent.

Proof: Let $x \in D$. Then

$$\forall \varepsilon > 0 : \quad \exists n_0 : \quad \forall n, m \geq n_0 : \quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon,$$

so $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, and therefore convergent. Define the pointwise limit $f : D \longrightarrow \mathbb{R}$ by

$$f(x) := \lim_{n \to \infty} f_n(x), \qquad x \in D.$$

We have to show $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and $x \in D$ arbitrary. Then

$$\exists n_0 : \quad \forall n, m \geq n_0 : \quad |f_n(x) - f_m(x)| < \varepsilon/2.$$

Choose such an n_0, fix $n \geq n_0$ and consider the limit $m \to \infty$. This gives

$$|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon.$$

As $x \in D$ is arbitrary, we get

$$\|f - f_n\|_\infty = \sup_{x \in D} |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon.$$

■

Proposition 3 *([K] 8.3.7, Dini's criterion)*
Let $[a, b]$ be a bounded closed interval, (f_n) a sequence of functions on $[a, b]$. Suppose:

(i) *$f_n \to f^*$ pointwise,*

(ii) *f_n continuous for all n, f^* continuous.*

(iii) *(f_n) is an increasing sequence, i.e.*

$$\forall x \in [a, b] : \quad \forall n, m \in \mathbb{N} : \quad n > m \;\Rightarrow\; f_n(x) \geq f_m(x).$$

Then $f_n \to f^*$ uniformly.

Proof: Because of (i) and (iii) we have $f_n(x) \leq f^*(x)$ for all $x \in [a, b]$. Assume the convergence is not uniform. Then

$$\exists \varepsilon > 0 : \quad \forall n \in \mathbb{N} : \quad \exists x_n \in [a, b] : \quad f_n(x_n) \leq f^*(x_n) - \varepsilon. \tag{2}$$

According to the Bolzano-Weierstrass theorem, the sequence (x_n) has an accumulation point x^*.

Fix $m \in \mathbb{N}$. The functions f_m and f^* are continuous, so

$$\begin{aligned}
&\exists \delta > 0 : \quad \forall x \in [a,b] : \\
&|x - x^*| < \delta \ \Rightarrow \big(|f^*(x) - f^*(x^*)| < \varepsilon/3 \ \wedge \ |f_m(x) - f_m(x^*)| < \varepsilon/3\big). \quad (3)
\end{aligned}$$

Choose such a δ and choose $k \in \mathbb{N}$ such that $k \geq m$ and $|x_k - x^*| < \delta$. (This is possible because x^* is an accumulation point of the sequence (x_n).) Then

$$f_m(x^*) \overset{(3)}{<} f_m(x_k) + \varepsilon/3 \overset{\text{(iii)}}{\leq} f_k(x_k) + \varepsilon/3 \overset{(2)}{\leq} f^*(x_k) - 2\varepsilon/3 \overset{(3)}{<} f^*(x^*) - \varepsilon/3.$$

Passing to the limit $m \to \infty$ gives $f^*(x^*) \leq f^*(x^*) - \varepsilon/3$. This is a contradiction. ■