Let (b_n) be a sequence in $\mathbb{R}$.

Proposition 1 *If* $\limsup b_n = L \in \mathbb{R}$ *then*

(i) L *is an accumulation point of* (b_n),

(ii) $\forall \varepsilon > 0: \quad \exists n_0 \in \mathbb{N}: \quad \forall n \geq n_0: \quad b_n < L + \varepsilon.$

Proof: (i) Let V be the set of accumulation points of (b_n). V is not empty because $\limsup b_n \in \mathbb{R}$. We reason step by step for $k = 1, 2, 3, \ldots$.

$$\exists B_k \in V: \quad L - 1/(2k) < B_k \leq L,$$

$$\exists n_k \in \mathbb{N}: \quad (k > 1 \ \Rightarrow\ n_k > n_{k-1}) \ \wedge\ |b_{n_k} - B_k| < 1/(2k).$$

So for each $k \in \mathbb{N}_+$

$$|b_{n_k} - L| \overset{\triangle}{\leq} |b_{n_k} - B_k| + |B_k - L| < 1/k.$$

Therefore $b_{n_k} \to L$ as $k \to \infty$.

(ii) Observe that (b_n) is bounded from above. (Why?) We argue by contradiction. Assume

$$\exists \varepsilon > 0: \quad \forall n \in \mathbb{N}: \quad \exists j \geq n: \quad b_j \geq L + \varepsilon.$$

Then (b_n) has a subsequence b_{n_k} with $b_{n_k} \geq L + \varepsilon$ for all k.

This subsequence is bounded (check!) and therefore, by the Bolzano-Weierstrass theorem, has an accumulation point $\tilde{b} \in V$ with $\tilde{b} \geq L + \varepsilon$. Then $L = \sup V \geq L + \varepsilon$, a contradiction. ■

Let

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{1}$$

be a (real) power series.

Proposition 2 *(Radius of convergence)*

(i) *If* $\limsup \sqrt[k]{|a_k|} = 0$ *then* (1) *is pointwise absolutely convergent on* $\mathbb{R}$.

(ii) *If* $\limsup \sqrt[k]{|a_k|} =: 1/R \in (0, \infty)$ *then* (1) *is pointwise absolutely convergent in* $(x_0 - R, x_0 + R)$ *and divergent for all* x *with* $|x - x_0| > R$.

(iii) *If* $\limsup \sqrt[k]{|a_k|} = +\infty$ *then* (1) *is convergent only for* $x = x_0$.

Proof: (i) Let $x \in \mathbb{R}$, w.l.o.g. $x \neq x_0$. We have $\sqrt[k]{|a_k|} \to 0$. (Check!). So

$$\sqrt[k]{|a_k|} < \frac{1}{2|x - x_0|}$$

for sufficiently large k. Root test for $\sum_{k=0}^{\infty} |a_k(x-x_0)^k|$:

$$\sqrt[k]{|a_k||x-x_0|^k} = \sqrt[k]{|a_k|}|x-x_0| < 1/2 < 1$$

for k sufficiently large. So (1) absolutely convergent.

(ii) 1. Let $x \in \mathbb{R}$ with $|x-x_0| < R$, w.l.o.g. $x \neq x_0$. Choose $\varepsilon \in (0, \frac{R}{|x-x_0|} - 1)$. Then $|x-x_0| < \frac{R}{1+\varepsilon}$ and because of Proposition 1 (ii)

$$\exists k_0 : \quad \forall k \geq k_0 : \quad \sqrt[k]{|a_k|} \leq \limsup \sqrt[k]{|a_k|} + \frac{\varepsilon}{2R} \leq \frac{1}{R}\left(1+\frac{\varepsilon}{2}\right).$$

Root test for $\sum_{k=0}^{\infty} |a_k(x-x_0)^k|$:

$$\sqrt[k]{|a_k||x-x_0|^k} = \sqrt[k]{|a_k|}|x-x_0| \leq \frac{1+\varepsilon/2}{1+\varepsilon} < 1$$

for k sufficiently large. So (1) absolutely convergent.

2. Let $x \in \mathbb{R}$ with $|x-x_0| > R$. Choose $\varepsilon \in (0, 1-\frac{R}{|x-x_0|})$. Then $|x-x_0| > \frac{R}{1-\varepsilon}$. Because of Proposition 1 (i) there is a subsequence (a_{k_j}) of (a_k) with $\sqrt[k_j]{|a_{k_j}|} \to 1/R$, so

$$\sqrt[k_j]{|a_{k_j}|} > \frac{1}{R}\left(1-\frac{\varepsilon}{2}\right)$$

for j sufficiently large. Therefore

$$\sqrt[k_j]{|a_{k_j}||x-x_)|^{k_j}} = \sqrt[k_j]{|a_{k_j}|}|x-x_0| > \frac{1-\varepsilon/2}{1-\varepsilon} > 1$$

and so

$$\neg(a_k|x-x_0|^k \to 0).$$

Consequently, (1) is divergent.

(iii) Let $x \in \mathbb{R} \setminus \{x_0\}$. The sequence $(\sqrt[k]{|a_k|})$ is unbounded above, and therefore it has a subsequence $(\sqrt[k_j]{|a_{k_j}|})$ going to infinity, so $\sqrt[k_j]{|a_{k_j}|}|x-x_0| > 1$ for all sufficiently large j.As in (ii) 2. this implies that (1) diverges. ■