

## The number $e$

Let

$$e_n := \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}_+.$$

**Definition and Theorem:** The sequence  $\{e_n\}$  is convergent, its limit is named  $e$ .

We have  $e = 2.718\dots$

Auxiliary propositions for the proof:

**1. Bernoulli's inequality:** Let  $n \in \mathbb{N}$  and  $x > -1$ . Then

$$(1+x)^n \geq 1+nx. \quad (1)$$

**Proof:** By induction: The statement is true for  $n = 0$ . Assume now

$$(1+x)^n \geq 1+nx \quad (2)$$

for some  $n \in \mathbb{N}$ . Then

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \stackrel{(2), x > -1}{\geq} (1+x)(1+nx) = 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x. \end{aligned}$$

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2. For each  $a \in \mathbb{R}$  we have

$$(1-a)(1+a+\dots+a^{n-1}) = 1-a^n. \quad (3)$$

(Check!)

**Proof of convergence of  $\{e_n\}$ :**

I.  $\{e_n\}$  is increasing:

$$\begin{aligned} \frac{e_{n+1}}{e_n} &= \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^n = \left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n} \\ &= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n} \stackrel{(1), x = -\frac{1}{(n+1)^2}}{\geq} \left(1 - \frac{1}{n+1}\right) \cdot \frac{n+1}{n} = 1. \end{aligned}$$

So  $e_{n+1} \geq e_n$ .

II.  $\{e_n\}$  is bounded above: Binomial theorem:

$$e_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^n \binom{n}{k} \frac{1}{n^k}$$

Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\dots(n-k+1)}{1 \cdot 2 \cdot \dots \cdot k}.$$

So

$$\binom{n}{k} \frac{1}{n^k} = \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n \cdot \dots \cdot n} \leq \frac{1}{k!} \leq \frac{1}{2^{k-1}}.$$

So

$$e_n \leq 1 + \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \stackrel{(3)}{=} 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} < 3.$$

I. and II.  $\Rightarrow \{e_n\}$  convergent. ■