Exercises Analysis 1 (2WA30) Lecture 1

1. Are the following subsets of $\mathbb{R}$ bounded above / below? Find supremum, infimum, maximum and minimum, if they exist. Give reasons for your answers.

$$\begin{aligned}
A &= \{x \in \mathbb{R} \,|\, \exists n \in \mathbb{N} : 2n-1 < x < 2n\}, \\
B &= \left\{ -\frac{1}{n} \,\middle|\, n \in \mathbb{N}_+ \right\}, \\
C &= \{x \in \mathbb{R} \,|\, 4x - x^2 > 3\}, \\
D &= \{x \in \mathbb{R} \,|\, 4x - x^2 \geq 3\}, \\
E &= [0,1] \backslash \mathbb{Q}
\end{aligned}$$

2. Let $A \subset \mathbb{R}$ be nonempty and bounded. Show:

a) For all $\varepsilon > 0$ there is an $x \in A$ such that $x > \sup A - \varepsilon$. (**Hint:** Give a proof by contradiction!)

b) Let z be an upper bound for A such that for all $\varepsilon > 0$ there is an $x \in A$ such that $x > z - \varepsilon$. Then $z = \sup A$.

c) Formulate the statements analogous to **a)** and **b)** for $\inf A$ (no proof required).

3. Let $A, B \subset \mathbb{R}$ be nonempty and bounded. Define

$$\begin{aligned}
-A &= \{-a \,|\, a \in A\}, \\
A + B &= \{a + b \,|\, a \in A,\, b \in B\}, \\
A - B &= \{a - b \,|\, a \in A,\, b \in B\}.
\end{aligned}$$

Show

a)

$$\begin{aligned}
\sup(-A) &= -\inf A, \\
\sup(A + B) &= \sup A + \sup B, \\
\inf(A - B) &= \inf A - \sup B,
\end{aligned}$$

b)

$$(\forall a \in A\; \forall b \in B :\; a < b) \;\Rightarrow\; \sup A \leq \inf B.$$

4. $\star$[1] The most straightforward way to represent real numbers is by "infinite decimal fractions" (IDFs), i.e. numbers of the form

[1]More challenging exercises for ambitious students are marked by $\star$ here and in the sequel. These exercises are not part of the regular homework and need not be handed in.

$$\pm a_0.a_1a_2a_3\ldots, \qquad a_0 \in \mathbb{N},\ a_i \in \{0,\ldots,9\}, i > 0$$

Check that the set of IDFs is completely ordered. More precisely:

Let A be a nonempty set of IDFs that is bounded above. Show that there is an IDF which is the supremum of A.

Hint: Show first that you can assume without loss of generality that A contains positive IDF's. Then try to find an algorithm which generates the digits of the supremum one by one.