Exercises Analysis 1 (2WA30) Lecture 3

1. Let $\{a_n\}$, $\{b_n\}$ be sequences with $a_n \to a^*$, $b_n \to b^*$, $a^*, b^* \in \mathbb{R}$. Let $c \in \mathbb{R}$. Show:

a) $ca_n \to ca^*$,

b) $a_n + b_n \to a^* + b^*$.

c) $\star$ Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$. Show:

$$\sqrt{a_n} \to \sqrt{a^*}.$$

(**Hint:** Distinguish the two cases $a^* = 0$ en $a^* > 0$. In the second case, use

$$\sqrt{a_n} - \sqrt{a^*} = \frac{a_n - a^*}{\sqrt{a_n} + \sqrt{a^*}}.)$$

2. Let $\{a_n\}$ be a sequence with $a_n \to \pm\infty$ [1], and $a_n \neq 0$ for all n. Let $\{b_n\}$ be a bounded sequence. Show:

a) $a_n + b_n \to \pm\infty$,

b) $\dfrac{b_n}{a_n} \to 0$.

3. Determine whether the following sequences converge and determine the limit or the improper limit if they exist. Indicate explicitly which rules and theorems you use.

$$a_n := \frac{1}{n^2} - \sqrt{n}, \qquad b_n := \frac{1}{\sqrt{n}} + (-1)^n n, \qquad c_n := \left(-\frac{3}{n}\right)^n,$$

$$d_n := \frac{(1.01)^n}{n^{2007}}, \qquad e_n := \frac{n^2 - 5n + 4}{n^3 + 8n^2 + 2}, \qquad f_n := \sqrt[n]{n^3},$$

$$g_n := \frac{3^n - n^3}{2^n + n^2 + 7}, \qquad h_n := \frac{\sqrt{n} - \frac{1}{2n}}{\sqrt[3]{n} + \frac{1}{n^2}}, \qquad k_n := \frac{n^8 + n^5}{4n^8 + 5n^6 + 3}$$

4. Let the sequence $\{a_n\}$ be given by

$$a_0 = 1, \quad a_{n+1} = a_n + \frac{1}{a_n} \quad n = 0, 1, 2, 3, \ldots$$

Determine whether $\{a_n\}$ converges and find the limit if it exists.

[1] This means: $a_n \to +\infty$ or $a_n \to -\infty$.