Exercises Analysis 1 (2WA30) Lecture 5

1. A series $\sum_{k=1}^{\infty} a_k$ with $a_k = b_k - b_{k-1}$, $k = 1, 2, \ldots$ is called *telescoping*. (More precisely: This name is used if one works with this representation.)

a) Show: A telescoping series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence (b_n) converges, and in this case

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} b_n - b_0.$$

b) Find with the help of a)

$$\sum_{k=2}^{\infty} \frac{2k+1}{k^2(k+1)^2}.$$

2. Kosmala Ex. 7.1.1 without (d),(h),(j),(k) (If no initial value for the summation index is given, start with $k = 1$.)

3. Let $\sum a_n$, $\sum b_n$ be two series, $c \in \mathbb{R}$. Are the following statements true? (Give a proof or a counterexample.)[1]

a) "If $\sum a_n$ and $\sum b_n$ are convergent then $\sum(a_n + b_n)$ is convergent as well, and $\sum(a_n + b_n) = \sum a_n + \sum b_n$."

b) "If $\sum a_n$ is convergent, then $\sum(ca_n)$ is convergent as well, and $\sum(ca_n) = c\sum a_n$."

c) "If $\sum a_n$ is convergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is divergent."

d) "If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is divergent as well."

4. Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Show:

$$\left|\sum_{k=1}^{\infty} a_k\right| \le \sum_{k=1}^{\infty} |a_k|.$$

(This is the "triangle inequality for infinitely many terms".)

5. Let $q \in \mathbb{R}$, $|q| < 1$. Find the value of the following series:

$$\sum_{k=7}^{\infty} \frac{q^k}{2}, \quad \sum_{k=7}^{\infty} \left(\frac{q}{2}\right)^k, \quad \sum_{k=0}^{\infty} \frac{2^{k/2}}{2^{2k}}, \quad \sum_{k=0}^{\infty} \frac{4^{k-5}}{5^{2k+2}}.$$

[1] Observe that it is unimportant at what value of n the summation starts.

6. $\star$[2] A rubber ball is dropped from height h_0 above a plane surface and falls down under the influence of gravity acceleration g. It bounces back up, falls down etc. In the k-th jump, the ball reaches maximum height $h_k = \alpha h_{k-1}$, $k = 1, 2, 3, \ldots$, with $\alpha \in (0, 1)$.

For how long does the ball move up and down in total?

[2] This problem is not difficult but involves some elementary mechanics which is beyond the scope of our course.