Exercises Analysis 1 (2WA30) Lecture 6

1. [K] Ex. 7.2.1. a,b,e,f,j,k,m

2. Let $\sum a_n$ be a series with positive terms. Show:

a) If
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$$
then $\sum a_n$ is convergent.

b) If
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$$
then $\sum a_n$ is divergent.

(**Hint:** Use [K] Theorem 7.3.3.)

3. Let $\sum a_n$ be a series with nonnegative terms. Show:

a) If
$$\lim_{n\to\infty} \sqrt[n]{a_n} = r < 1$$
then $\sum a_n$ is convergent.

b) If
$$\lim_{n\to\infty} \sqrt[n]{a_n} = r > 1$$
then $\sum a_n$ is divergent.

(**Hint:** Use [K] Theorem 7.3.8.)

4. [K] Ex. 7.3.7. a,b,e,g,i

5. $\star$ (Dirichlet's test)

Let (a_n) be a decreasing sequence in $\mathbb{R}$ with $a_n \to 0$. Let (b_n) be a sequence in $\mathbb{R}$ and let (s_n), given by $s_0 = 0$, $s_n = b_1 + \ldots + b_n$ $(n > 0)$, the corresponding sequence of partial sums. Suppose (s_n) is bounded.

a) Show: The series $\sum_{n=0}^{\infty} s_n(a_{n+1} - a_n)$ is absolutely convergent.

b) Check that for all $N > 0$ we have
$$\sum_{n=1}^{N} a_n(s_n - s_{n-1}) + \sum_{n=1}^{N} s_n(a_{n+1} - a_n) = s_N a_{N+1}.$$

c) Show: the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent. (Use a) and b).)

d) Leibniz' criterion (except for the estimate) can directly be concluded from c). How?