Exercises Analysis 1 (2WA30) Lecture 7

1. Find out whether the following series converge:

$$\sum_{n=1}^{\infty} \frac{4n^2+2n+3}{n^3+\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{4n^2+2n+3}{n^3\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/5}}, \quad \sum_{n=1}^{\infty} (-1)^n(\sqrt[n]{n}-1)$$

2. Give examples for two series $\sum a_k$, $\sum b_k$ such that

$$a_k \to 0, \quad \frac{a_{k+1}}{a_k} \to 1, \quad \sum a_k \text{ convergent}$$

and

$$b_k \to 0, \quad \frac{b_{k+1}}{b_k} \to 1, \quad \sum b_k \text{ divergent.}$$

3. Show:

a) For all $x, y \in \mathbb{R}$ with $x < y$ we have $\exp(x) < \exp(y)$.

b) For each polynomial p given by $p(x) = \sum_{k=0}^{n} a_k x^k$ there is an $M \in \mathbb{R}$ such that

$$\forall x > M: \quad \exp(x) \geq p(x).$$

("The exponential function grows faster than any polynomial.")

c) $\lim_{n\to\infty} \exp(-n) = 0$.

4. Find out whether the following series converge:

$$\sum_{n=1}^{\infty} \exp(-n), \quad \sum_{n=1}^{\infty} n^2 \exp(-\sqrt{n})$$

5. $\star$ Let $\sum a_n$ be a divergent series with $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = 0$. Show: For each real $s > 0$ there is a subsequence (a_{n_k}) of (a_n) such that

$$\sum_{k=1}^{\infty} a_{n_k} = s.$$