Exercises Analysis 1 (2WA30) Lecture 8

1. Let $D \subset \mathbb{R}$ be unbounded from above, $a \in (D\cap(-\infty,a))'$, $b \in (D\cap(b,\infty))'$, $L \in \mathbb{R}$, $f : D \longrightarrow \mathbb{R}$.

Give the definitions of the following limit statements. Give an equivalent characterization in terms of sequences and show that it is equivalent.

a) $\lim\limits_{x\uparrow a} f(x) = L$,

b) $\lim\limits_{x\downarrow b} f(x) = \infty$,

c) $\lim\limits_{x\to\infty} f(x) = -\infty$,

2. Let $D \subset \mathbb{R}$, $a \in (D \cap (-\infty,a))' \cup (D \cap (a,\infty))'$, $f : D \longrightarrow \mathbb{R}$.

Show: $\lim\limits_{x\to a} f(x)$ exists if and only if: The one-sided limits $\lim\limits_{x\uparrow a} f(x)$ and $\lim\limits_{x\downarrow a} f(x)$ both exist and

$$\lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x).$$

In this case

$$\lim_{x\to a} f(x) = \lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x)$$

3. Let $D \subset \mathbb{R}$, $a \in D'$, $f, g : D \longrightarrow \mathbb{R}$, f bounded, and $\lim\limits_{x\to a} g(x) = +\infty$. Show

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

4. Let $I \subset \mathbb{R}$ be an interval, $f, g : I \longrightarrow \mathbb{R}$ continuous and $g(x) \neq 0$ for all $x \in I$.

Show:

a) We have

$$(\forall x \in I : \ g(x) > 0) \ \vee \ (\forall x \in I : \ g(x) < 0)\,.$$

b) the functions (defined by pointwise operations)

$$f + g,\ f \cdot g,\ \frac{f}{g} : \ I \longrightarrow \mathbb{R}$$

are continous on I.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous periodic function with period $T > 0$. Show: There is an $x \in \mathbb{R}$ such that $f(x) = f(x + T/2)$.

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, $f(0) = 1$, and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0.$$

Show: there is an $x^* \in \mathbb{R}$ such that $f(x^*) = \max\{f(x), | \, x \in \mathbb{R}\}$.