Exercises Analysis 1 (2WA30) Lecture 9

1. Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$, $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = 1/x, \qquad g(x) = \begin{cases} x & (x \in \mathbb{Q}) \\ 0 & (x \notin \mathbb{Q}) \end{cases}, \qquad h(x) = \begin{cases} x^2 & (x \in \mathbb{Q}) \\ 0 & (x \notin \mathbb{Q}) \end{cases}$$

In which points are these functions differentiable? Find the derivatives in these points.

2. Kosmala Ex. 5.3.15 c,d,f

3. Let $I \subset \mathbb{R}$ be an interval. A differentiable function $F : I \longrightarrow \mathbb{R}$ is called a *primitive* for the function $f : I \longrightarrow \mathbb{R}$ (on I) if $F'(x) = f(x)$ for all $x \in I$.

Show: If F_1 and F_2 are two primitives for f on I then there is a constant $C \in \mathbb{R}$ such that $F_2 \equiv F_1 + C$, i.e. $F_2(x) = F_1(x) + C$ for all $x \in I$.

4. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be given by

$$f(x) = \ln(x).$$

a) Give the n-th order Taylor polynomial T_n for f around $a = 1$ and give a representation for the remainder term $R_n(x)$ if $\ln(x)$ is approximated by $T_n(x)$.

b) Show that $R_n(x) \to 0$ as $n \to \infty$ for all $x \in [1, 2]$.

c) Find the value of the infinite series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

a) Show inductively that f is k times differentiable and

$$f^{(k)}(x) = \begin{cases} p_k(\frac{1}{x})e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

with some polynomial p_k.

b) Give the Taylor polynomial for f of degree $n \in \mathbb{N}$ around 0. For which $x \in \mathbb{R}$ does the Taylor series of f around 0 converge? Does it converge to $f(x)$?

6. $\star$ Using Mathematica, calculate high order Taylor polynomials for your own chosen, infinitely differentiable function. Make a conjecture about the interval on which the Taylor polynomials converge to the function.