Exercises Analysis 1 (2WA30) Lecture 12

1. Kosmala Ex. 8.4.17

2. Let the sequence of functions (f_n), $n \in \mathbb{N}_+$, on $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} 0 & \text{if } |x| > 1/n, \\ nx + 1 & \text{if } x \in [-1/n, 0), \\ 1 - nx & \text{if } x \in [0, 1/n]. \end{cases}$$

a) Find the pointwise limit function $f^* : \mathbb{R} \longrightarrow \mathbb{R}$.

b) Does the sequence of functions (f_n) converge uniformly to f^* ?

c) Show that the function series

$$\sum_{n=1}^{\infty} 2^{-n} f_n(x - \tfrac{1}{n})$$

converges pointwise on $\mathbb{R}$.

d) Prove: the function $s : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$s(x) := \sum_{n=1}^{\infty} 2^{-n} f_n(x - \tfrac{1}{n}), \qquad x \in \mathbb{R},$$

is continuous.

3. ($\star$) Let $m \in \mathbb{R}$ and let (a_k) be a sequence of real number such that

$$\sum_{k=0}^{\infty} |a_k| e^{-mk}$$

converges. Show: The function $s : [m, \infty) \longrightarrow \mathbb{R}$ defined by

$$s(x) = \sum_{k=0}^{\infty} a_k e^{-kx}$$

is infinitely differentiable on (m, ∞).

4. For each of the following power series, find all $x \in \mathbb{R}$ for which they converge:

$$\sum_{k=1}^{\infty} \frac{x^k}{2^k k}, \quad \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \quad \sum_{k=1}^{\infty} k\, 3^k x^k, \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k.$$