Exercises Analysis 1 (2WA30) Lecture 14

1. (The Identity theorem for power series under weaker assumptions)

a) Let $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with positive radius of convergence. Show: If $f(x_n) = 0$ for a sequence $\{x_n\}$ with $x_n \to x_0$, $x_n \neq x_0$, then $a_k = 0$ for all $k \in \mathbb{N}$.
Hint: For $j \in \mathbb{N}$, define

$$f_{(j)}(x) := \sum_{k=0}^{\infty} a_{j+k}(x-x_0)^k$$

and show by induction over j: For all $j \in \mathbb{N}$:

$$f_{(j)}(x_n) = 0 \text{ for all } n \in \mathbb{N} \text{ and } \quad a_j = 0.$$

b) Conclude: If

$$g(x) = \sum_{k=0}^{\infty} b_k(x-x_0)^k, \quad h(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$$

are two power series with positive radii of convergence and $g(x_n) = h(x_n)$ for a sequence (x_n) with $x_n \to x_0$, $x_n \neq x_0$, then $b_k = c_k$ for all $k \in \mathbb{N}$.

2. Find closed expressions for the functions given by the power series

$$\sum_{k=0}^{\infty} k^3(x-2)^k, \quad \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!}.$$

3. Find coefficients a_k, $k \in \mathbb{N}$, such that the function f given by the power series

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

satisfies the equations

a) $xf''(x) - xf'(x) - f(x) = 0, \quad f'(0) = 1,$

b) $\star$ $f''(x) = 2f(x)f'(x), \quad f(0) = f'(0) = 1.$

Find the radius of convergence of the power series and a representation of f in terms of elementary functions.

4. $\star$ Let

$$f(x) := \sum_{k=0}^{\infty} a_k x^k, \qquad g(x) := \sum_{l=0}^{\infty} b_l x^l$$

be two functions defined by their power series with radii of convergence $R_1 > 0,\ R_2 > 0$, respectively.

Show that their product h (given by $h(x) := f(x)g(x)$) has a power series representation with radius of convergence $R \geq \min(R_1, R_2)$. Give the coefficients of the power series representing h.