A bounded set in $\mathbb{Q}$ without supremum

The following example shows that the field $\mathbb{Q}$ is not completely ordered, i.e. there are bounded subsets of $\mathbb{Q}$ without supremum (in $\mathbb{Q}$!).

As an example we choose

$$A := \{x \in \mathbb{Q} \,|\, x > 0 \,\wedge\, x^2 < 2\}.$$

1. Clearly $1 \in A \neq \varnothing$, furthermore A is bounded above, as for $x > 2$ we have $x > 0$ and $x^2 > 4 > 2$, so 2 is an upper bound for A.

2. Assume $c \in \mathbb{Q}$ is the supremum of A. Then $c \geq 1 > 0$. We show:

$$\forall\, \xi \in \mathbb{Q}: \quad 0 < \xi < c \quad \Longrightarrow \quad \xi \in A. \tag{1}$$

From $\xi < c = \sup A$ it follows that ξ is not an upper bound for A, therefore there is $\xi' > \xi$ with $\xi' \in A$, so ${\xi'}^2 < 2$ and therefore $\xi^2 < {\xi'}^2 < 2$, so $\xi \in A$.

3. Define

$$\xi := \frac{2c+2}{c+2}.$$

Then certainly $\xi > 0$ and $\xi \in \mathbb{Q}$. It is easily checked that

$$\xi = c - \frac{c^2-2}{c+2}, \tag{2}$$

$$\xi^2 = 2 + \frac{2(c^2-2)}{(c+2)^2}. \tag{3}$$

4. As there is no rational number z with $z^2 = 2$, one of the two strict inequalities $c^2 < 2$ or $c^2 > 2$ must hold. In both cases, we will derive a contradiction.

$$\text{I.} \quad c^2 < 2 \quad \left.\begin{array}{l} \overset{(2)}{\Longrightarrow} \xi > c \Longrightarrow \xi \notin A \Longrightarrow \xi^2 > 2 \\ \overset{(3)}{\Longrightarrow} \xi^2 < 2 \end{array}\right\} \text{contradiction.}$$

$$\text{II.} \quad c^2 > 2 \quad \left.\begin{array}{l} \overset{(2)}{\Longrightarrow} \xi < c \overset{(1)}{\Longrightarrow} \xi \in A \Longrightarrow \xi^2 < 2 \\ \overset{(3)}{\Longrightarrow} \xi^2 > 2 \end{array}\right\} \text{contradiction.}$$

Therefore, the set A can have no rational supremum. ∎