Products of absolutely convergent series

Let

$$\sum_{k=0}^{\infty} a_k, \qquad \sum_{l=0}^{\infty} b_l$$

be two series.

Define for $n \in \mathbb{N}$

$$c_n := \sum_{k+l=n} a_k b_l = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0.$$

The series $\sum_{n=0}^{\infty} c_n$ is called the **Cauchy product** of the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{l=0}^{\infty} b_l$.

Motivation: "multiplying termwise and summing up:", as in finite sums:

	a_0	a_1	a_2	a_3	$\ldots$
b_0	$a_0 b_0$	$a_1 b_0$	$a_2 b_0$	$a_3 b_0$	$\ldots$
b_1	$a_0 b_1$	$a_1 b_1$	$a_2 b_1$	$\ldots$	
b_2	$a_0 b_2$	$a_1 b_2$	$\ddots$		
b_3	$a_0 b_3$	$\vdots$			
$\vdots$	$\vdots$				

The term c_n contains the products on the n-th rising diagonal in this table.

Proposition 1: Assume that

$$\sum_{k=0}^{\infty} a_k = A, \qquad \sum_{l=0}^{\infty} b_l = B$$

are absolutely convergent. Then the Cauchy product is absolutely convergent as well, and

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof: I. Absolute convergence of $\sum c_n$:

Let $\sigma_N := \sum_{n=0}^{N} |c_n|$. So the sequence (σ_N) is increasing. Moreover,

$$\begin{aligned} \sigma_N &\leq \sum_{n=0}^{N} \sum_{k+l=n} |a_k||b_l| = \sum_{k+l \leq N} |a_k||b_l| \leq \sum_{k \leq N,\, l \leq N} |a_k||b_l| \\ &= \sum_{k=0}^{N} |a_k| \sum_{l=0}^{N} |b_l| \leq \sum_{k=0}^{\infty} |a_k| \sum_{l=0}^{\infty} |b_l|. \end{aligned}$$

So (σ_N) bounded above $\Rightarrow \sum |c_n|$ convergent $\Rightarrow \sum c_n$ absolutely convergent.

II. Convergence to AB: Define the set of index pairs

$$I_N := \{(k,l) \mid k+l \leq 2N \wedge (k > N \vee l > N)\}.$$

Then

$$\left|\sum_{n=0}^{2N} c_n - \sum_{k=0}^{N} a_k \sum_{l=0}^{N} b_l\right| = \left|\sum_{(k,l)\in I_N} a_k b_l\right|$$

$$\leq \sum_{(k,l)\in I_N} |a_k||b_l| = \sum_{k=0}^{2N} |a_k| \sum_{l=0}^{2N} |b_l| - \sum_{k=0}^{N} |a_k| \sum_{l=0}^{N} |b_l| \overset{N\to\infty}{\longrightarrow} 0.$$

So

$$\sum_{n=0}^{\infty} c_n = \lim_{N\to\infty} \sum_{n=0}^{2N} c_n = \lim_{N\to\infty} \left(\sum_{k=0}^{N} a_k \sum_{l=0}^{N} b_l\right) = AB.$$

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The exponential function

Lemma: For all $x \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is absolutely convergent.

Proof: Without loss of generality $x \neq 0$. Let $a_k := \frac{x^k}{k!}$. Then

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|}{k} \overset{k\to\infty}{\longrightarrow} 0.$$

Ratio test: $\sum |a_k|$ convergent $\Rightarrow \sum a_k$ absolutely convergent. ■

Definition: Let the function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Proposition 2:

I. $\exp(1) = e$

II. $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.

Proof: I. Let

$$e_n := \left(1 + \frac{1}{n}\right)^n, \quad e'_n := \sum_{k=0}^{n} \frac{x^k}{k!}.$$

By definition: $e_n \to e$, $e'_n \to \exp(1)$ als $n \to \infty$.

a) Binomial theorem:

$$\begin{aligned} e_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{n \cdot \ldots \cdot n} \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} = e'_n. \end{aligned}$$

Limit $n \to \infty$: $e \leq \exp(1)$.

b) Choose $m \in \mathbb{N}_+$ en $n \geq m$ arbitrary. Then

$$\begin{aligned} e_n &= 1 + \sum_{k=1}^{n} \frac{1}{k!} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{n^k} \\ &\geq 1 + \sum_{k=1}^{m} \frac{1}{k!} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{n^k} \\ &= 1 + \sum_{k=1}^{m} \frac{1}{k!} \cdot 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-k+1}{n}. \end{aligned}$$

m fixed, $n \to \infty$:

$$e \geq 1 + \sum_{k=1}^{m} \frac{1}{k!} = e'_m.$$

Now $m \to \infty$: $e \geq \exp(1)$.

a) and b) together imply I.

II. The series

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \exp(y) = \sum_{l=0}^{\infty} \frac{y^l}{l!}$$

are absolutely convergent, according to the lemma. Proposition 1:

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} c_n,$$

where

$$c_n = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{1}{n!}(x+y)^n.$$

So

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y).$$

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Reminder: If $a > 0$ and $p \in \mathbb{Q}$ met $p = m/n$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_+$, then by definition

$$a^p := \sqrt[n]{a^m}.$$

Proposition 3: For all rational numbers p and all $x \in \mathbb{R}$ we have

$$\exp(px) = \exp(x)^p.$$

Proof: I. $\forall x \in \mathbb{R},\ n \in \mathbb{N}:\ exp(nx) = \exp(x)^n$.

Proof by induction: Statement is true for $n = 0$ (Check!). Assume $\exp(nx) = \exp(x)^n$ for an $n \in \mathbb{N}$. Then

$$\exp((n+1)x) = \exp(nx + x) = \exp(nx)\exp(x) = \exp(x)^n \exp(x) = \exp(x)^{n+1}.$$

II. $\exp(-x) = \exp(x)^{-1}$ because of

$$\exp(x)\exp(-x) = \exp(0) = 1.$$

III. $\forall x \in \mathbb{R},\ n \in \mathbb{N}_+:\ \exp(x/n) = \exp(x)^{1/n}$ because of

$$\exp(x) = \exp\left(n\frac{x}{n}\right) \stackrel{\text{I}}{=} \exp\left(\frac{x}{n}\right)^n.$$

IV. Let now in general $p = m/n$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}_+$. Without loss of generality $p \neq 0$. Distinguish two cases:
a) $m > 0$:

$$\exp\left(\frac{m}{n}x\right) \stackrel{\text{I}}{=} \exp\left(\frac{x}{n}\right)^m \stackrel{\text{III}}{=} \left(\exp(x)^{1/n}\right)^m = \exp(x)^{m/n}.$$

b) $m < 0$

$$\exp\left(\frac{m}{n}x\right) = \exp\left(-\frac{-m}{n}x\right) \stackrel{\text{II}}{=} \exp\left(\frac{-m}{n}x\right)^{-1} \stackrel{\text{a)}}{=} \left(\exp(x)^{-m/n}\right)^{-1} = \exp(x)^{m/n}.$$

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Corollary: For all $x \in \mathbb{Q}$ we have

$$\exp(x) = e^x.$$

(Check!)