Let $I \subset \mathbb{R}$ be an interval, $a \in I$, and $f : I \longrightarrow \mathbb{R}$ be an infinitely differentiable function, i.e. a function for which the derivatives $f^{(k)}$ of all orders $k \in \mathbb{N}$ exist on I. Let $x \in \mathbb{R}$, and consider the sequence $(T_n(x))$ of n-th order Taylor approximations of f around a, evaluated at x. By definition of series convergence, this sequence converges for $n \to \infty$ if and only if the series

$$T_\infty(x) := \lim_{n\to\infty} T_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{1}$$

is convergent. It is called the **Taylor series** of f around a.

Two questions arise:

- For which $x \in \mathbb{R}$ does (1) converge?
- If (1) converges and $x \in I$, do we have $T_\infty(x) = f(x)$?

A partial answer is given by the following observation:

Proposition: Let $x \in I$, and let $R_n(x)$ be the remainder term in Taylor's theorem (as given in the lecture). Then

$$\text{The Taylor series (1) converges to } f(x) \quad \Longleftrightarrow \quad R_n(x) \overset{n\to\infty}{\longrightarrow} 0.$$

(Check!)