TECHNISCHE UNIVERSITEIT EINDHOVEN

Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA31), Friday 31 januari 2014, 9.00-12.00 uur.

General instruction: Give reasons and arguments for all statements you make! Theorems that have been covered in the course may be used without reproving them. However, their use should be mentioned clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let A and B be two nonempty bounded subsets of $\mathbb{R}$ with the following property:

For all $b \in B$ there is a sequence (a_n) of elements in A such that $a_n \to b$ as $n \to \infty$.

Show that $\sup B \leq \sup A$. Which inequality holds between $\inf A$ and $\inf B$ (no proof required)?

2. Let (a_n), $n = 1, 2, 3, \ldots$, be the sequence in $\mathbb{R}$ given by

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} = \sum_{k=n}^{2n} \frac{1}{k}.$$

Show: (a_n) is convergent.

3. Let (a_n) be a bounded sequence in $\mathbb{R}$ with precisely two accumulation points, namely 1 and 2, and let (b_n) be a bounded sequence in $\mathbb{R}$ with precisely two accumulation points, namely 2 en 4. Let (c_n) be the sequence in $\mathbb{R}$ given by $c_n = a_n b_n$. Let V be the set of all accumulation points of (c_n). Show: $V \neq \varnothing$ en $V \subseteq \{2, 4, 8\}$.

4. **a)** Show: For all real $x > 0$ we have

$$1 - \cos(x) \leq \frac{x^2}{2} + \frac{x^3}{6}$$

(**Hint:** Use (for example) Taylor's theorem with remainder.)

b) Find out whether the series

$$\sum_{n=1}^{\infty} (1 - \cos(1/n))$$

converges. (**Hint:** Use **a)**.)

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5. Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be given and let the sequence (a_n) in $\mathbb{R}$ be given by $a_n = f(\sqrt{n})$, $n \in \mathbb{N}$.

- **a)** Show: If $\lim_{x\to\infty} f(x) = L$ with $L \in \mathbb{R}$ then also $\lim_{n\to\infty} a_n = L$.
- **b)** Give an example of a function f such that $\lim_{n\to\infty} a_n = 0$, but $\lim_{x\to\infty} f(x) = 0$ does not hold.
- **c)** Show: If f is decreasing and $\lim_{n\to\infty} a_n = 0$, then $\lim_{x\to\infty} f(x) = 0$.

6. Let the sequence of functions (f_n) on $\mathbb{R}$ be given by

$$f_n(x) = \frac{\sin(nx)}{1+nx^2}, \quad x \in \mathbb{R},\ n = 1,2,3,\ldots.$$

- **a)** Show: (f_n) is pointwise convergent on $\mathbb{R}$.
- **b)** Show: (f_n) is not uniformly convergent on $\mathbb{R}$.
- **c)** Fix $a > 0$. Show: (f_n) is uniformly convergent on $[a,\infty)$.
- **d)** Show: the function series
$$\sum_{n=1}^{\infty} \frac{f_n}{n} \tag{1}$$
is uniformly convergent on $[a,\infty)$
- **e)** Show: Eqn. (1) defines a continuous function $s : (0,\infty) \longrightarrow \mathbb{R}$.

7. Find coefficients a_k, $k \in \mathbb{N}$, such that the function f given by the power series

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

satisfies the equations

$$f'''(x) = f(x), \quad f(0) = 1,\ f'(0) = f''(0) = 0.$$

Find the radius of convergence of this power series.

The following numbers of point can be earned per problem:

1 : 6	2 : 5	3 : 6	4a : 3	5a : 2	6a : 1	7 : 6
			4b : 2	5b : 2	6b : 1	
				5c : 2	6c : 1	
					6d : 2	
					6e : 1	

The grade is determined by dividing the total number of points by 4 and rouding to one decimal.