TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Final test Analyse 1 (2WA31), 29 Januari 2016, 9.00-12.00 uur.

General instruction: Give reasons and arguments for all statements you make. Indicate clearly where you use previously shown results.

For this test no material (books, notes, calculators, computers etc.) is allowed.

1. Let (z_n) be the sequence given by

$$z_n = (-1)^n + \frac{1}{n}, \quad n \in \mathbb{N}_+ = \{1, 2, 3, \ldots\}.$$

Let $A \subset \mathbb{R}$ be a subset with the properties

$$\inf A = -2, \quad \sup A = 1.$$

Let

$$B = \{a z_n \mid a \in A, n \in \mathbb{N}_+\}.$$

Find $\sup B$ and prove your result.

2. Let (a_n) be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = a^* > 0$. Determine whether the following sequences converge:

a) $b_n = \max\{a_n, a_{n+7}\}$, **b)** $c_n = \max\{a_n, a_{n+1} + a_{n+2}\}$.

(Give a proof or a counterexample. In case of convergence, give the limit.)

3. We say that a pair of sequences (a_n) and (b_n) in $\mathbb{R}$ has property (V) if they are bounded and 0 is an accumulation point for the sequence (d_n) given by $d_n = a_n - b_n$.

a) Give an example of a pair of (a_n) and (b_n) that has property (V) such that 1 is an accumulation point of (a_n) but not of (b_n).

b) Show: If two sequences (a_n) and (b_n) have property (V) then they have a common accumulation point.

4. Determine whether the following series converge:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^{2n} + (2n)^n}}$, **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p_n^2}$.

Here $(p_n) = (2, 3, 5, 7, 11, \ldots)$ is the sequence of the prime numbers.

5. Zij $a, b \in \mathbb{R}$, $a < b$, $f : (a, b] \longrightarrow \mathbb{R}$.

a) Give the definition of the limit property

$$\lim_{x \to a} f(x) = -\infty. \tag{1}$$

b) Show: If f is continuous on $(a, b]$ and satisfies (1) then f is bounded from above.

c) Show under the same assumptions: f takes its maximal value on $(a, b]$, i.e. there is an $x^* \in (a, b]$ such that

$$f(x^*) = \max\{f(x) | x \in (a, b]\}.$$

6. Let (f_n) be the sequence of functions on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \dfrac{x}{n^2} & \text{if } x \in (0, n^2), \\ 1 & \text{if } x \geq n^2. \end{cases}$$

(**Hint:** Sketch the graphs of $f_1, f_2, \ldots$.)

a) Show that the sequence (f_n) converges pointwise on $\mathbb{R}$ and give the limit function.

b) Determine whether (f_n) converges uniformly on $\mathbb{R}$.

c) Show that the series of functions $\sum_{n=1}^{\infty} f_n$ is pointwise convergent on $\mathbb{R}$.

d) Show that the series of functions $\sum_{n=1}^{\infty} f_n$ is not uniformly convergent on $\mathbb{R}$. (**Hint:** Argue by contradiction.)

e) Show: For each $a > 0$ the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[0, a]$.

f) Show: The function $s : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$s(x) = \sum_{n=1}^{\infty} f_n(x), \qquad x \in \mathbb{R},$$

is continuous.

7. Find coefficients a_k, $k \in \mathbb{N}$, such that the function f given by the power series

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

satisfies the equations

$$x f''(x) - x f'(x) - f(x) = 0, \quad f'(0) = 1.$$

Determine the radius of convergence of the power series and find a closed representation of f in terms of known functions.

The following numbers of points can be obtained per problem:

1 : 5	2a : 2	3a : 2	4a : 3	5a : 2	6a : 1	7 : 6
	2b : 4	3b : 3	4b : 3	5b : 2	6b : 1	
				5c : 2	6c : 1	
					6d : 1	
					6e : 1	
					6f : 1	

The grade is determined by dividing the total number of points obtained by 4 and round to one decimal.