TECHNISCHE UNIVERSITEIT EINDHOVEN

Faculteit Wiskunde en Informatica

Final Test Analysis 1 (2WA31), 15 April 2016, 18.00-21.00 uur.

General instruction: Give reasons and arguments for all statements that you make. Indicate clearly where you use results that have been proved in the course.

No materials (books, notes, calculators computers etc.) are allowed.

1. Let A, B two nonempty, bounded subsets of $\mathbb{R}$ with the following properties:

(i) $\forall a \in A\ \forall b \in B: \quad a < b$,

(ii) $\forall \varepsilon > 0: \ \exists a \in A \ \ \exists b \in B: \quad |a - b| < \varepsilon$.

Show: $\sup A = \inf B$.

2. For each sequence (a_k) of positive real numbers we define the sequence (p_n) by

$$p_1 = a_1, \quad p_{n+1} = a_{n+1} p_n, \qquad \text{i.e. } p_n = a_1 a_2 \ldots a_n.$$

a) Show: If (p_n) converges to a number $P \in (0, \infty)$ then $\lim\limits_{k \to \infty} a_k = 1$.

b) Give an example of a sequence (a_k) of positive real numbers such that

$$\lim_{k \to \infty} a_k = 1 \quad \text{and} \quad \lim_{n \to \infty} p_n = 0.$$

c) Give an example of a sequence (a_k) of positive real numbers such that

$$\lim_{k \to \infty} a_k = 1 \quad \text{and} \quad \lim_{n \to \infty} p_n = +\infty.$$

(**Hint:** Consider $\ln p_n$.)

3. Let (a_n) and (b_n) be two bounded sequences in $\mathbb{R}$. Suppose that 1 en 3 are the only accumulation points of (a_n), and that 0 en 2 are the only accumulation points of (b_n).

Show: There is a n_0 such that $|a_n - b_n| > 1/2$ for all $n \geq n_0$.

4. **a)** Give an example of two convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.

b) Show: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n b_n$ is also convergent.

please turn over

5. Let $b \in \mathbb{R}$, $f : (-\infty, b) \longrightarrow \mathbb{R}$ with the following properties:

(i) $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to b} f(x) = 1$,

(ii) f is continuous

(iii) f is injective, i.e. if $f(x_1) = f(x_2)$ for $x_1, x_2 \in (-\infty, b)$ then $x_1 = x_2$.

a) Give the definitions of the properties in (i).

b) Show: The image of f is the interval $(-\infty, 1)$.

c) Show: f is strictly increasing.

6. Let (f_n) be the sequence of functions on $\mathbb{R}$ given by

$$f_n(x) = \arctan(nx), \qquad x \in \mathbb{R}.$$

a) Show that (f_n) is pointwise convergent on $\mathbb{R}$ and give the limit function f^*.

b) Show that the convergence is not uniform on $\mathbb{R}$.

c) Show that the convergence (of the restrictions) is uniform on $(1, \infty)$.

d) Is the convergence of the restrictions uniform on $(0, \infty)$?

e) Show that the function series

$$s(x) = \sum_{n=1}^{\infty} \frac{f_n}{n^3}$$

defines a continuously differentiable function $s : \mathbb{R} \longrightarrow \mathbb{R}$.

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the conditions

$$(1 - x^2) f''(x) - 4x f'(x) - 2f(x) = 0, \qquad f(0) = 1, \quad f'(0) = 0.$$

Find the radius of convergence of this power series and give a closed expression for f in terms of elementary functions.

The following points can be obtained per problem:

1 : 5	2a : 2	3 : 5	4a : 2	5a : 2	6a : 1	7 : 6
	2b : 2		4b : 4	5b : 2	6b : 1	
	2c : 2			5c : 2	6c : 1	
					6d : 1	
					6e : 2	

The mark is determined by dividing the number of points by 4 and round to one decimal.