TECHNISCHE UNIVERSITEIT EINDHOVEN
Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA31), February 3, 2017, 9.00-12.00 uur.

General instruction: Give reasons and arguments for all statements you make! Theorems that have been covered in the course may be used without reproving them. However, their use should be mentioned clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let $A \subset \mathbb{R}$ be bounded from below and nonempty. Furthermore, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function, i.e.

$$\forall\, x_1, x_2 \in \mathbb{R} : \quad x_1 < x_2 \;\Rightarrow\; f(x_1) \le f(x_2).$$

Define $f(A) := \{f(x) \,|\, x \in A\}$.

- **a)** Show: $\sup f(A) \le f(\sup A)$.
- **b)** Give an example for A and f that satisfy the above conditions such that $\sup f(A) < f(\sup A)$.
- **c)** Show: If f is (moreover) continuous then $\sup f(A) = f(\sup A)$.

2. **a)** For $a > 0$ let

$$x_n := \left(a + \frac{1}{n}\right)^n.$$

For all $a > 0$, investigate whether the sequence (x_n) converges and if it does, find the limit.

b) Let the sequence (a_n) be given by

$$a_0 = 1, \qquad a_{n+1} = a_n + \frac{1}{\sqrt{a_n}(n+1)^2}.$$

Show that (a_n) converges.
(**Hint:** First consider the sequence (b_n) given by

$$b_0 = 1, \qquad b_{n+1} = b_n + \frac{1}{(n+1)^2}. \;)$$

3. Let (a_n), (b_n), and (c_n) be three bounded sequences in $\mathbb{R}$ with the following properties:

(i) $a_n \le c_n \le b_n$ for all $n \in \mathbb{N}$,
(ii) both (a_n) and (b_n) have precisely the accumulation points 0 and 1.

Show:

- **a)** All accumulation points of (c_n) are in the interval $[0, 1]$.
- **b)** The numbers 0 and 1 are accumulation points of (c_n).

please turn over

4. Investigate whether the following series converge:

$$\textbf{a)} \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}, \qquad \textbf{b)} \sum_{n=2}^{\infty} \frac{\sqrt[3]{n+2}}{n\sqrt{n-1}}$$

c) Is the following statement true? (Give a proof or a counterexample.)
"Let $a_n > 0$ for all n and assume that $\sum_{n=1}^{\infty} a_n$ is divergent. Let furthermore (b_n) be a sequence with $b_n \to 1$. The $\sum_{n=1}^{\infty} a_n b_n$ is divergent as well."

5. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a continuous function with the properties

$$\lim_{x \to 0} f(x) = +\infty, \qquad \lim_{x \to \infty} f(x) = 2.$$

a) Give the definition of these limit relations.

b) Show: If there is a $x_0 \in (0, \infty)$ such that $f(x_0) < 2$ then

$$\exists\, x_1, x_2 \in (0, \infty) : \quad x_1 \neq x_2 \ \wedge \ f(x_1) = f(x_2).$$

6. Let (a_n) be a sequence in $\mathbb{R}$ such that $\sum_{n=0}^{\infty} |a_n|$ is convergent.

a) Show: By

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(x^n), \qquad x \in \mathbb{R},$$

a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined. This function is continuous.

b) Assume now that $\sum_{n=0}^{\infty} n|a_n|$ is convergent and show that f is differentiable in $(-1, 1)$.

c) Let $c > 0$. Assume $a_n \neq 0$ for all $n \in \mathbb{N}$. Give conditions on (a_n) such that f is differentiable in $(-c, c)$.

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the conditions

$$x^2 f''(x) - 2xf'(x) - (x^2 - 2)f(x) = 0, \qquad f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0.$$

Find the radius of convergence of this power series and a closed expression for f.

The following numbers of point can be earned per problem:

1a : 1	2a : 3	3a : 3	4a : 2	5a : 2	6a : 2	7 : 6
1b : 2	2b : 3	3b : 3	4b : 2	5b : 3	6b : 2	
1c : 2			4c : 3		6c : 1	

The grade is determined by dividing the total number of points by 4 and rounding to one decimal.