EINDHOVEN UNIVERSTY OF TECHNOLOGY

Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA30), February 2, 2018, 9.00-12.00 uur.

General instruction: Give reasons and arguments for all statements you make! Theorems and results that have been covered in the course and the homework may be used without reproving them. However, their use should be indicated clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let A and B be nonempty and bounded subsets of $(0, \infty)$ such that $\sup(A) = \inf(B)$. Let

$$C = \left\{ \frac{b}{a} \mid b \in B,\ a \in A \right\}.$$

Find $\inf(C)$ and prove your statement.

2. Investigate whether the following recursively defined sequences are convergent, and if so, find the limit.

a) (a_n) given by $a_1 = 1$, $a_n = a_{n-1}\left(1 + \frac{1}{\sqrt{n}}\right)$ $(n = 2, 3, \ldots)$;

b) (b_n) given by $b_0 = 0$, $b_n = \left(b_{n-1} + \frac{1}{4}\right)^2$ $(n = 1, 2, 3, \ldots)$.

(**Hint:** Show first that $b_n \in [0, 1/4]$ for all $n \in \mathbb{N}$.)

3. Let (a_n) be a bounded sequence and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Are the following statements true (for all such (a_n) and f)? For each of them, give a proof or a counterexample.

a) "If v is an accumulation point of (a_n) then $f(v)$ is an accumulation point of $(f(a_n))$".

b) "If w is an accumulation point of $(f(a_n))$ then there is an accumulation point v of (a_n) such that $w = f(v)$."

c) "If w is an accumulation point of $(f(a_n))$ and v is a number such that $w = f(v)$ then v is an accumulation point of (a_n)".

4. **a)** Let (a_n) be a bounded sequence. Show that the series

$$\sum_{n=0}^{\infty} \frac{a_n^n}{n!}$$

is convergent.

b) For which $p \in \mathbb{Q}$ does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2} - \sqrt{n}}{n^p}$$

converge? (Give a complete answer, i.e., consider all possible p.)

please turn over

5. Let $f : (0, 2) \longrightarrow \mathbb{R}$ such that

$$\lim_{x \downarrow 1} f(x) = +\infty.$$

a) Give the definition of this limit property.

b) Show that

$$\lim_{z \to \infty} f\left(\frac{z+1}{z}\right) = +\infty.$$

c) Let $g : (0, 2) \longrightarrow \mathbb{R}$ be continuous. Show: If g takes its minimum value in some $x_0 \in (0, 2)$ then g is not injective.

6. **a)** Show that the function series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3 x} \tag{1}$$

converges pointwise on $[0, \infty)$.

b) Show that the function $s : [0, \infty) \longrightarrow \mathbb{R}$ defined by (1) is continuous.

c) Show: For any $a > 0$, s is differentiable on the interval $[a, \infty)$. Calculate the derivative (in terms of a function series).

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the conditions

$$x f''(x) + 2 f'(x) + x f(x) = 0, \quad f(0) = 1.$$

Find the radius of convergence of this power series and a closed expression for f.

The following numbers of point can be earned per problem:

1 : 5	2a : 3	3 : 5	4a : 3	5a : 1	6a : 1	7 : 6
	2b : 4		4b : 3	5b : 2	6b : 2	
				5c : 2	6c : 3	

The grade is determined by dividing the total number of points by 4 and rounding to one decimal.