EINDHOVEN UNIVERSTY OF TECHNOLOGY

Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA30), April 11, 2018, 18.00-21.00 uur.

General instruction: Give reasons and arguments for all statements you make! Theorems and results that have been covered in the course and the homework may be used without reproving them. However, their use should be indicated clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let $C \subset \mathbb{R}$ be nonempty and let z be a lower bound for C. Show: $z = \inf C$ if and only if there is a sequence (c_n) in C such that $\lim\limits_{n\to\infty} c_n = z$.

2. Let (a_n) be a sequence in $\mathbb{R}$ such that

$$|a_{n+1} - a_n| \leq \frac{1}{n^2}, \qquad n = 1, 2, \ldots.$$

a) Show that (a_n) is bounded.

b) Show that (a_n) is convergent.

3. Let A and B be two nonempty subsets of $\mathbb{R}$ such that

$$\inf\{|a - b| \mid a \in A, b \in B\} = 0$$

and A is bounded. Show that there are convergent sequences (a_n) in A and (b_n) in B with

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

(**Hint:** Use **1.**)

4. **a)** Is the following statement true? (Give a proof or a counterexample.)
"If $\sum_k a_k$ and $\sum_k b_k$ are convergent series in $\mathbb{R}$ then $\sum_k a_k b_k$ is also convergent."

b) Find the radius of convergence of the power series

$$\sum_{k=0}^{\infty} (3x)^{(k^2)}.$$

please turn over

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5. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a continuous function such that

$$\lim_{x \to 0} f(x) = +\infty, \qquad \lim_{x \to +\infty} f(x) = 2.$$

a) Give the definition of these limit properties.

b) Show that f is bounded below.

c) Let $R(f)$ denote the image of f. Show:

$$(\inf_{x \in (0,\infty)} f(x), +\infty) \subset R(f).$$

6. **a)** Show that the function series

$$s(x) := \sum_{n=1}^{\infty} x^n \sin(nx) \tag{1}$$

converges pointwise on $(-1, 1)$.

b) Let $a \in (0, 1)$. Show that this function series converges uniformly on $[-a, a]$.

c) Show that the function $s : (-1, 1) \longrightarrow \mathbb{R}$ defined by (1) is continuous.

d) Let $a \in (0, 1)$. Show that s is differentiable on $[-a, a]$, and give a series representation for s'.

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the conditions

$$f''(x) - 2xf'(x) - 2f(x) = 2, \quad f(0) = 0, \quad f'(0) = 0.$$

Find the radius of convergence of this power series and a closed expression for f.

The following numbers of point can be earned per problem:

1 : 4	2a : 3	3 : 5	4a : 3	5a : 2	6a : 1	7 : 6
	2b : 3		4b : 3	5b : 2	6b : 2	
				5c : 2	6c : 2	
					6d : 2	

The grade is determined by dividing the total number of points by 4 and rounding to one decimal.