EINDHOVEN UNIVERSTY OF TECHNOLOGY
Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA30), February 1, 2019, 9:00-12:00.

General instruction: Give reasons and arguments for all statements you make! Theorems and results that have been covered in the course and the homework may be used without reproving them. However, their use should be indicated clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let A and B be bounded, nonempty subsets of $\mathbb{R}$ such that

(i) $\forall\, a \in A,\, b \in B: \quad a - b > 1$

(ii) There is a sequence (a_n) in A and a sequence (b_n) in B such that $\lim\limits_{n\to\infty}(a_n - b_n) = 1$.

Show: $\inf A - \sup B = 1$.

2. For sequences (x_n), (y_n) in $(0,\infty)$ we define the relation $\sim$ by

$$(x_n) \sim (y_n) \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{x_n}{y_n} = 1.$$

Are the following statements true for all sequences (a_n), (b_n), (c_n) in $(0,\infty)$? Give a proof or a counterexample.

a) "If $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$ then $(a_n) \sim (c_n)$."

b) "If $(a_n) \sim (b_n)$ then $\lim_{n\to\infty}(a_n - b_n) = 0$."

c) "If $(a_n) \sim (b_n)$ then $(a_n + 1) \sim (b_n + 1)$." (**Hint:** Rewrite $\frac{a_n+1}{b_n+1} - 1$.)

3. Let (a_n) be a bounded sequence in $\mathbb{R}$ such that all its accumulation points lie in the interval $(0,1)$. Show that there is an $n_0 \in \mathbb{N}$ such that $a_n \in (0,1)$ for all $n \geq n_0$.

(**Hint:** Argue by contradiction.)

4. Investigate whether the following series converge:

$$\textbf{a)} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 1}}, \qquad \textbf{b)} \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$$

c) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n!\, x^{(n^2)}.$$

5. Let $f : (-\infty, 0) \longrightarrow \mathbb{R}$ be a differentiable function with the properties

$$\text{(i)} \quad \lim_{x\to-\infty} f(x) = 1, \qquad \text{(ii)} \quad \lim_{x\to 0} f(x) = +\infty.$$

a) Give the definition of these properties.

b) Let $R(f)$ denote the range of f. Show $(1,\infty) \subset R(f)$.

c) Show that the derivative of f is not bounded on $(-\infty, 0)$.

please turn over

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6. Let $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Assume that the function and its derivative are bounded on $\mathbb{R}$.

a) Give an example of such a function Φ that is not constant.

b) Show: The function series

$$\sum_{k=0}^{\infty} \Phi(kx)e^{-k}$$

is pointwise convergent on $\mathbb{R}$ and therefore defines a function $s : \mathbb{R} \longrightarrow \mathbb{R}$ by $s(x) = \sum_{k=0}^{\infty} \Phi(kx)e^{-k}$, $x \in \mathbb{R}$.

c) Show that s is continuous on $\mathbb{R}$.

d) Show that s is differentiable on $\mathbb{R}$.

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the differential equation

$$(1 + x^2)f''(x) + 4xf'(x) + 2f(x) = 0$$

with additional conditions

$$f(0) = 1, \quad f'(0) = 0.$$

Find the radius of convergence of this power series and a closed expression for f.

The following numbers of point can be earned per problem:

1 : 5	2a : 2	3 : 5	4a : 2	5a : 2	6a : 1	7 : 6
	2b : 2		4b : 2	5b : 2	6b : 1	
	2c : 2		4c : 2	5c : 2	6c : 2	
					6d : 2	

The grade is determined by dividing the total number of points by 4 and rounding to one decimal.