EINDHOVEN UNIVERSITY OF TECHNOLOGY
Dept. of Mathematics and Computer Science

Final test Analysis 1 (2WA30), April 13, 2019, 13:30-16:30.

General instruction: Give reasons and arguments for all statements you make! Theorems and results that have been covered in the course and the homework may be used without reproving them. However, their use should be indicated clearly and explicitly.

Use of any material (books, notes, calculators, computers etc.) is not allowed.

1. Let (V_k) be a sequence of nonempty subsets of $\mathbb{R}$ such that

$$V_1 \subset V_2 \subset V_3 \subset \ldots \subset V_k \subset V_{k+1} \subset \ldots$$

and their union $V := \bigcup_{k=1}^{\infty} V_k$ is bounded above. Let $s_k := \sup V_k$. Show that (s_k) is convergent and

$$\lim_{k\to\infty} s_k = \sup V.$$

2. Let (a_n), (b_n) be the sequences in $\mathbb{R}$ given by the recursive definitions

$$a_0 = 0, \quad a_{n+1} = \frac{a_n^2+1}{2}, \qquad b_0 = 2, \quad b_{n+1} = \frac{b_n^2+1}{2}.$$

Investigate whether these sequences converge, and in this case, find their limit.

3. Let (a_n) be a bounded sequence in $\mathbb{R}$ and let the sequence (b_n) be defined by $b_n = a_n + a_{n+1}$. Let B be an accumulation point of (b_n). Show that there are two (possibly identical) accumulation points A_1 and A_2 of (a_n) such that $B = A_1 + A_2$.

4. **a)** Does the series

$$\sum_{n=2}^{\infty} \sqrt{\frac{n^2-2n+5}{n^3+6}}$$

converge? Prove your answer.

b) Let (a_n) and (b_n) be two sequences in $\mathbb{R}$ such that both $\sum_n a_n^2$ and $\sum_n b_n^2$ converge. Show that $\sum_n a_n b_n$ converges absolutely.

c) Let

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

be a power series such that the power series $\sum_{k=0}^{\infty} a_{2k} x^{2k}$ has convergence radius R_0 and the power series $\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$ has convergence radius R_1. Find the convergence radius R of the power series (1) in terms of R_0 and R_1. Prove the result.

please turn over

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5. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be three functions with the following properties:

(i) $\lim_{x\to\infty} f(x) = L$,

(ii) $\lim_{t\to 2} g(t) = +\infty$

(iii) h is continuous and periodic, i.e. there is a $T > 0$ such that

$$h(x+T) = h(x), \qquad x \in \mathbb{R}.$$

a) Give the definitions of the properties (i) and (ii).

b) Show: The range of h is a bounded, closed interval, i.e. there are $c, d \in \mathbb{R}$ such that $R(h) = [c, d]$.

c) Show that

$$\lim_{t\to 2} f(g(t) + h(t)) = L.$$

6. **a)** Show that for all $z \in \mathbb{R}$

$$|\sin(z)| \leq |z|.$$

b) Show that the function series

$$\sum_{n=0}^{\infty} \sin(x^n) \tag{2}$$

converges pointwise on $(-1, 1)$.

c) Show that for all $a \in (0, 1)$, the function series (2) converges uniformly on $[-a, a]$.

d) Let $s : (-1, 1) \longrightarrow \mathbb{R}$ be the function defined by (2). Show that s is continuous on $(-1, 1)$.

e) Show that s is differentiable on $(-1, 1)$.

7. Find coefficients a_n, $n \in \mathbb{N}$, such that the function f given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies the differential equation

$$xf''(x) - f'(x) + 4x^3 f(x) = 0$$

with additional conditions

$$f(0) = 1, \quad f''(0) = 0.$$

Find the radius of convergence of this power series and a closed expression for f.

The following numbers of point can be earned per problem:

1 : 5	2 : 5	3 : 5	4a : 2	5a : 2	6a : 1	7 : 6
			4b : 2	5b : 2	6b : 1	
			4c : 2	5c : 2	6c : 1	
					6d : 2	
					6e : 2	

The grade is determined by dividing the total number of points by 4 and rounding to one decimal.