Are the following statements true? Give a proof or a counterexample.

Sets of rational and real numbers

- "Let $A \subset \mathbb{R}$ be bounded and nonempty. If $\inf A < x < \sup A$ then $x \in A$."
- "Let $A \subset \mathbb{R}$ be bounded and nonempty. Then $\inf A < \sup A$."
- "Let $A \subset \mathbb{R}$ be bounded and nonempty. Then $\inf A \leq \sup A$. "

⋆ "If F is any proper ordered subfield of $\mathbb{R}$ that contains $\mathbb{Q}$ (with the ordering from $\mathbb{R}$) then F is not completely ordered."

Sequences of numbers

- "Any increasing sequence is bounded below."
- "Any decreasing sequence has at most one accumulation point."
- "Every sequence in $\mathbb{R}$ has a subsequence that is decreasing or increasing."
- "Let (a_n) be a sequence in $\{0, 1\}$ which takes both these values infinitely often. Then each sequence in $\{0, 1\}$ is a subsequence of (a_n)."
- "Any sequence in $\mathbb{R}$ has at most finitely many accumulation points."
- "If (x_n) is a sequence in (a, b) then all its accumulation points are in (a, b)."
- "If (a_n) and (b_n) are two sequences in $\mathbb{R}$, $a_n \leq b_n$ for all $n \in \mathbb{N}$, A is an accumulation point of (a_n), and B is an accumulation point of (b_n) then $A \leq B$."
- "If (a_n) and (b_n) are two sequences in $\mathbb{R}$, A is an accumulation point of (a_n), and B is an accumulation point of (b_n) then $A+B$ is an accumulation point of $(a_n + b_n)$."

⋆ "There is no sequence in $\mathbb{R}$ whose accumulation points are precisely the irrational numbers."

Series of numbers

- "If $\sum a_n$ is convergent and $b_n \leq a_n$ for all $n \in \mathbb{N}$ then $\sum b_n$ is convergent."

⋆ "If $\sum a_n$ is convergent then $\sum a_n^3$ is convergent."

⋆ "If $\sum a_n$ is convergent and (b_n) is a sequence with $b_n \to 1$ then $\sum a_n b_n$ is convergent."

- "If $\sum a_n$ is an alternating series that satisfies the conditions of the Leibniz criterion then $\sum a_n$ is absolutely convergent."

- “If $\sum a_n$ is absolutely convergent then $\sum(-1)^n|a_n|$ is convergent.”
- “If $\sum a_n$ is absolutely convergent then $\sum(-1)^n|a_n|$ satisfies the conditions of the Leibniz criterion, and is therefore convergent.”

⋆ “If $\sum a_n$ is conditionally convergent and $s \in \mathbb{R}$ arbitrary then there are infinitely many different reorderings of $\sum a_n$ with value s.”

Limits of functions, continuity, differentiability

- “If $f : (a, b) \longrightarrow \mathbb{R}$ is bounded and $x_0 \in (a, b)$ then the one-sided limits $\lim_{x\uparrow x_0} f(x)$ and $\lim_{x\downarrow x_0} f(x)$ exist.”
- “If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $\lim_{x\to-\infty} f(x) = +\infty$, and $\lim_{x\to+\infty} f(x) = -\infty$ then f is surjective.”
- “For $x \in (0, 1)$ we have $\arctan x < x < \tan x$.”
- “If $f : (-1, 1) \longrightarrow \mathbb{R}$ is differentiable everywhere then $\lim_{x\to 0} f'(x) = f'(0)$.”

Sequences and series of functions

- “If a series (f_n) of continuous functions converges pointwise to a continuous function f on an bounded and closed interval $[a, b]$ then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.”
- “If a series (f_n) of continuous functions converges uniformly to a continuous function f on an bounded and closed interval $[a, b]$ then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.”
- “If a sequence/series of functions is uniformly convergent on some domain D then it is also uniformly convergent on every subdomain $\tilde{D} \subset D$.”
- “If a sequence/series of functions is uniformly convergent on two subdomains D_1 and D_2 then it is also uniformly convergent on their union $D_1 \cup D_2$.
- “If a sequence/series of functions is uniformly convergent on infinitely many subdomains D_α, $\alpha \in I$, then it is also uniformly convergent on their union $\bigcup_{\alpha\in I} D_\alpha$.”
- “Functions that can be represented as uniformly convergent series whose terms are infinitely differentiable, are infinitely differentiable.”
- “If two power series $\sum a_n x^n$ and $\sum b_n x^n$ have radius of convergence R_1 and R_2, respectively, then their sum has convergence radius $\min\{R_1, R_2\}$.”

- "If two power series $\sum a_n x^n$ and $\sum b_n x^n$ have radius of convergence R_1 and R_2, respectively, then their (Cauchy) product has convergence radius $\min\{R_1, R_2\}$."

⋆ "The Taylor series of an infinitely differentiable functions has positive radius of convergence."