Solutions to final test Analysis 1 (2WA31)
February 2017

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- No rights can be derived from these solutions.
- Comparing your own solutions to the ones given here does not necessarily yield sufficient information on the former's correctness, as the problems often can be solved in various, different-looking ways.
- **Caution:** Reading these solutions can lead to an over-optimistic estimation of your abilities. It is no substitute to (trying to) solve the problems independently!

1. **a)** Let $y \in f(A)$. There is an $x \in A$ with $y = f(x)$, and we have $x \leq \sup A$. Therefore $y = f(x) \leq f(\sup A)$. So $f(\sup A)$ is an upper bound for $f(A)$, and, as $\sup f(A)$ is the least upper bound, $\sup f(A) \leq f(\sup A)$.

b) $A = (-1, 0)$, f given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then $\sup f(A) = \sup\{0\} = 0$, $f(\sup A) = f(0) = 1$.

c) Let $\varepsilon > 0$. As f is continuous at $\sup A$, there is a $\delta > 0$ such that

$$\forall \xi \in (\sup A - \delta, \sup A + \delta) : \quad f(\xi) > f(\sup A) - \varepsilon.$$

Furthermore, there is an $x \in A$ such that $x > \sup A - \delta$. So $x \in (\sup A - \delta, \sup A + \delta)$, and therefore $f(x) > f(\sup A) - \varepsilon$. So $f(\sup A) - \varepsilon$ is no upper bound for $f(A)$, and, as this holds for all $\varepsilon > 0$, $f(\sup A) \leq \sup f(A)$. The results follow from this and **a)**.

2. **a)** We distinguish three cases:
I. $a < 1$: Then

$$a + \frac{1}{n} < \frac{a+1}{2} < 1$$

for n large, and so

$$0 < x_n < \left(\frac{a+1}{2}\right)^n$$

for these n. Now, as $\left(\frac{a+1}{2}\right)^n \to 0$ (standard limit), we get $x_n \to 0$ by the squeeze theorem.

II. $a = 1$:

$$\left(1 + \frac{1}{n}\right)^n \to e$$

(standard limit).

III. $a > 1$: Then

$$a + \frac{1}{n} > \frac{a+1}{2} > 1$$

for n large, and so

$$x_n > \left(\frac{a+1}{2}\right)^n$$

for these n. Therefore $x_n \to +\infty$ because of $\left(\frac{a+1}{2}\right)^n \to +\infty$ (standard limit).

b) We have

$$b_n = \sum_{k=1}^{n+1} \frac{1}{k^2},$$

so (b_n) is the sequence of the partial sums of the convergent hyperharmonic series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Further, a simple induction argument shows that $1 \leq a_n \leq b_n$ for all n, so (a_n) has the upper bound $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Therefore, as (a_n) is also increasing, (a_n) is convergent.

3. a) Let v be an accumulation point of (c_n). Then there is a subsequence (c_{n_k}) such that $c_{n_k} \to v$ as $k \to \infty$. Suppose $v < 0$. Then $c_{n_k} < \frac{v}{2} < 0$ for k large, and so also $a_{n_k} < \frac{v}{2} < 0$. By the Bolzano-Weierstrass theorem, (a_{n_k}) has a convergent subsequence $(a_{n_{k_l}})$. Its limit w satisfies $w \leq \frac{v}{2} < 0$. This is not possible as $w \in \{0, 1\}$ because w is also an accumulation point of (a_n). Therefore $v \geq 0$.

Analogously, $v \leq 1$. (Fill in the details!)

b) As 0 is an accumulation point of (b_n) there is a subsequence (b_{n_k}) with $b_{n_k} \to 0$ as $k \to \infty$. By the Bolzano-Weierstrass theorem, (a_{n_k}) has a convergent subsequence $(a_{n_{k_l}})$. Its limit is either 0 or 1 because it is an accumulation point of (a_n). However, because of $a_{n_{k_l}} \leq b_{n_{k_l}} < \frac{1}{2}$ for l large, the second possibility is excluded, and therefore $a_{n_{k_l}} \to 0$ as $l \to \infty$. Now because of

$$a_{n_{k_l}} \leq c_{n_{k_l}} \leq b_{n_{k_l}},$$

we get from the squeeze theorem that $c_{n_{k_l}} \to 0$ as $l \to \infty$, and therefore 0 is an accumulation point of (c_n).

Analogously, one shows that 1 is an accumulation point of (c_n). (Fill in the details!)

4. a) The series is alternating, and $\frac{\ln n}{n} \to 0$ as $n \to \infty$ (standard limit). Setting

$$f(x) = \frac{\ln x}{x} \qquad (x > 0)$$

and calculating

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \qquad \text{for } x > e$$

we find that the sequence $\left(\frac{\ln n}{n}\right)$ is decreasing for $n \geq 3$. Therefore, convergence follows from the Leibniz criterion.

b) Observe first that the series has positive terms and

$$\frac{\sqrt[3]{n+2}}{n\sqrt{n-1}} \sim n^{\frac{1}{3}-\frac{3}{2}} = n^{-\frac{7}{6}}$$

for n large. More precisely,

$$\lim_{n\to\infty} \frac{\sqrt[3]{n+2}}{n\sqrt{n-1}} n^{\frac{7}{6}} = 1.$$

(Give the details!)
This implies

$$\frac{\sqrt[3]{n+2}}{n\sqrt{n-1}} \leq \frac{2}{n^{\frac{7}{6}}}$$

for n large, and therefore $\sum_{n=n_0}^{\infty} \frac{2}{n^{\frac{7}{6}}}$ is a convergent majorant (hyperharmonic series) for $\sum_{n=n_0}^{\infty} \frac{\sqrt[3]{n+2}}{n\sqrt{n-1}}$ for n_0 large. So $\sum_{n=2}^{\infty} \frac{\sqrt[3]{n+2}}{n\sqrt{n-1}}$ is convergent.

c) We have $b_n > \frac{1}{2}$ for $n \geq n_0$ with some $n_0 \in \mathbb{N}$. Therefore, the series $\sum_{n=n_0}^{\infty} a_n b_n$ has the divergent minorant $\sum_{n=n_0}^{\infty} \frac{a_n}{2}$. (Both have positive terms.) So $\sum_{n=1}^{\infty} a_n b_n$ is divergent.

5. a)

$$\forall\, M \in \mathbb{R}: \quad \exists\, \delta > 0: \quad x \in (0, \delta) \;\Rightarrow\; f(x) > M, \tag{1}$$

$$\forall\, \varepsilon > 0: \quad \exists\, z \in (0, \infty): \quad x > z \;\Rightarrow\; |f(x) - 2| < \varepsilon. \tag{2}$$

b) Let $y = \frac{f(x_0)+2}{2}$. Choosing $M = y$ in (1) shows

$$\exists\, \delta^* > 0: \quad x \in (0, \delta^*) \;\Rightarrow\; f(x) > y.$$

Fix $\xi_1 \in (0, \min(\delta^*, x_0))$. Then $f(\xi_1) > y > f(x_0)$. Applying the Intermediate Value theorem to f on the interval $[\xi_1, x_0]$ yields that there is an $x_1 \in (\xi_1, x_0)$ such that $f(x_1) = y$.
Choosing $\varepsilon = 2 - y > 0$ in (2) shows

$$\exists\, z^* \in (0, \infty): \quad x > z^* \;\Rightarrow\; f(x) > y.$$

(Check!) Fix $\xi_2 > \max(x_0, z^*)$. Then $f(x_0) < y < f(\xi_2)$. Applying the Intermediate Value theorem to f on the interval $[x_0, \xi_2]$ yields that there is an $x_2 \in (x_0, \xi_2)$ such that $f(x_2) = y = f(x_1)$.
The statement follows, as $x_1 < x_0 < x_2$.

6. **a)** Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f_n(x) = a_n \sin(x^n)$. Then $\|f_n\|_\infty = |a_n|$, and the given series of functions is uniformly convergent on $\mathbb{R}$ by the Weierstrass criterion. As this implies pointwise convergence, the function f is well-defined. Moreover, as uniform convergence preserves continuity and all f_n are continuous, f is also continuous.

b) Consider the series of functions

$$\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \cos(x^n).$$

Due to $\sup_{x\in[-1,1]} |f_n'(x)| = |na_n|$, the Weierstrass criterion yields the uniform convergence of this series of functions. From this and the result from **a)**, it follows that f is differentiable on $(-1, 1)$.

c) Observe that $\sup_{x\in[-c,c]} |f_n'(x)| = c^{n-1}n|a_n|$. So by the same reasoning as above, we get convergence of $\sum c^{n-1}n|a_n|$ (or, equivalently,$\sum c^n n|a_n|$) as sufficient condition for differentiability on $(-c, c)$.

7. Applying standard rules for calculations with power series and shifting the index we find

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} + 2\sum_{n=0}^{\infty} a_n x^n &= \\
\sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n + 2\sum_{n=0}^{\infty} a_n x^n &= 0.
\end{aligned}$$

Applying the identity theorem for power series ("comparison of coefficients") we find $a_0 = 0$ (as given already), and

$$a_{n-2} = (n(n-1) - 2n + 2)a_n = (n-1)(n-2)a_n$$

for $n \geq 2$, or

$$a_{n+2} = \frac{a_n}{n(n+1)}$$

for $n \geq 1$. It follows from $f'(0) = 1$ that $a_1 = 1$ and from $f''(0) = 0$ that $a_2 = 0$. It is easily seen (and proved by induction) now that $a_{2k} = 0$ and $a_{2k+1} = \frac{1}{(2k)!}$, so

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k)!}.$$

Applying e.g. the ratio test it is easy to see that the series converges for all $x \in \mathbb{R}$. Finally,

$$f(x) = x\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = x \cosh x.$$