Solutions to final test Analysis 1 (2WA30)
February 2019

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- No rights can be derived from these solutions.
- Comparing your own solutions to the ones given here does not necessarily yield sufficient information on the former's correctness, as the problems often can be solved in various, different-looking ways.
- **Caution:** Reading these solutions can lead to an over-optimistic estimation of your abilities. It is no substitute to (trying to) solve the problems independently!

1. For all $a \in A$, $b \in B$ we have $a > b+1$, so $b+1$ is a lower bound for A. Hence $\inf A \geq b+1$ for all $b \in B$, and therefore $\inf A - 1$ is an upper bound for B. Hence $\inf A - 1 \geq \sup B$, or equivalently $\inf A - \sup B \geq 1$.

Let $\varepsilon > 0$ be arbitrary. There is an $n \in \mathbb{N}$ such that $a_n - b_n < 1 + \varepsilon$, and therefore

$$\inf A - \sup B \leq a_n - b_n < 1 + \varepsilon.$$

As this estimate holds for all $\varepsilon > 0$, this implies $\inf A - \sup B \leq 1$. So $\inf A - \sup B = 1$.

2. **a)** The statement is true. Assume $(a_n) \sim (b_n)$, $(b_n) \sim (c_n)$. Then, by known calculation rules for limits of products

$$\frac{a_n}{c_n} = \underbrace{\frac{a_n}{b_n}}_{\to 1} \underbrace{\frac{b_n}{c_n}}_{\to 1} \to 1.$$

b) The statement is false. A possible counterexample is $a_n = n+1$, $b_n = n$.

c) The statement is true. Assume $(a_n) \sim (b_n)$. Then

$$0 \leq \left| \frac{a_n+1}{b_n+1} - 1 \right| = \left| \frac{a_n - b_n}{b_n+1} \right| = \frac{\left| \frac{a_n}{b_n} - 1 \right|}{1 + \frac{1}{b_n}} \leq \left| \frac{a_n}{b_n} - 1 \right| \to 0,$$

and therefore, by the Squeeze Theorem, $\frac{a_n+1}{b_n+1} - 1 \to 0$, which implies $(a_n+1) \sim (b_n+1)$.

3. Suppose the negation of the statement that is to be shown:

$$\forall\, n \in \mathbb{N}: \quad \exists\, m \geq n: \quad a_m \notin (0,1).$$

Therefore, there exists a subsequence (a_{n_k}) of (a_n) such that $(a_{n_k}) \notin (0,1)$. By the Bolzano-Weiserstrass theorem, this subsequence has a convergent subsequence $(a_{n_{k_l}})$. For its limit z we either have $z \geq 1$ or $z \leq 0$. In both cases, this is a contradiction to our assumption as z is also an accumulation point of the original sequence (a_n).

4. **a)** As

$$\lim_{n\to\infty} \sqrt{\frac{n^3}{n^3-1}} = 1,$$

we have for n large enough (say for $n \geq n_0$) that $\sqrt{\frac{n^3}{n^3-1}} < 2$ and therefore for these n

$$\frac{1}{\sqrt{n^3-1}} \leq \frac{2}{n^{3/2}}$$

As (up to a factor 2 and finitely many missing terms) the series $\sum_{n=n_0}^{\infty} \frac{2}{n^{3/2}}$ is a hypergeometric series with exponent $\frac{3}{2} > 1$, and therefore convergent, we can apply the majorant criterion to establish that also $\sum_{n=n_0}^{\infty} \frac{1}{\sqrt{n^3-1}}$, and hence $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$ is convergent.

b) We have

$$\frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = 2\left(\frac{n}{n+1}\right)^n \to \frac{2}{e} < 1$$

as $n \to \infty$. Therefore, the ratio test yields convergence of the series.

c) Calculating the ratio

$$\frac{(n+1)!|x|^{((n+1)^2)}}{n!|x|^{(n^2)}} = (n+1)|x|^{2n+1}$$

shows that the given power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. So the radius of convergence is 1.

5. **a)** (i): $\forall\, \varepsilon > 0 : \exists\, x_0 < 0 : x \leq x_0 \;\Rightarrow\; |f(x) - 1| < \varepsilon$.
(ii): $\forall\, M \in \mathbb{R} : \exists\, \delta > 0 : x \in (-\delta, 0) \;\Rightarrow\; f(x) > M$.

b) Let $y > 1$ be fixed. From (ii) with $M = y$ we get that there is a $\delta > 0$ such that $f(x) > y$ for all $x \in (-\delta, 0)$. Fix such a δ, and fix $x_2 \in (-\delta, 0)$. From (i) with $\varepsilon = y - 1$ we get that there is an x_0 such that $f(x) < y$ for all $x \leq x_0$. Fix such an x_0, and fix $x_1 < \min\{-\delta, x_0\}$ Then $x_1 < x_2$ and $f(x_2) < y$. As f is continuous (because f is differentiable) we can apply the Intermediate Value theorem on the interval $[x_1, x_2]$ and find that there is a $c \in (x_1, x_2)$ with $f(c) = y$, hence $y \in R(f)$.

c) By the Mean Value theorem we have that for all $x \in (-1, 0)$ there is a $\xi \in (-1, x)$ such that

$$f(x) = f(-1) + f'(\xi)(x+1).$$

Suppose f' would be bounded, say $|f'(z)| \leq L$ for all $z \in (-\infty, 0)$. This would imply

$$f(x) \leq f(-1) + L(x+1) \leq f(-1) + L, \qquad \text{for all } x \in (-1, 0),$$

which contradicts (ii). Therefore f' is unbounded.

6. **b)** Let M be an upper bound for $\{|\Phi(z)| \mid z \in \mathbb{R}\}$. Fix $x \in \mathbb{R}$. Then $|\Phi(kx)e^{-k}| \leq Me^{-k}$, and as $\sum_{k=0}^{\infty} Me^{-k}$ is convergent it is a convergent majorant for $\sum_{k=0}^{\infty} |\Phi(kx)e^{-k}|$, so the series $\sum_{k=0}^{\infty} \Phi(kx)e^{-k}$ is absolutely convergent.

c) Define $f_k : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_k(x) = \Phi(kx)e^{-k}$, $x \in \mathbb{R}$. Then, by the same arguments as in a), $\|f_k\|_\infty \leq Me^{-k}$ and uniform convergence of $\sum f_k$ is ensured by the Weierstrass M-test.

d) Let L be an upper bound for $\{|\Phi'(z)| \,|\, z \in \mathbb{R}\}$. Then $\|f_k'\|_\infty \le Lke^{-k}$, and, as $\sum ke^{-k}$ is convergent (why?) we get by the Weierstrass M-test that $\sum f_k'$ is uniformly convergent. Together with the convergence of $\sum f_k$ this implies by a known theorem that s differentiable.

7. The additional conditions imply $a_0 = 1$, $a_1 = 0$. Inserting the ansatz in the equation yields

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

and after rearranging and shifting the index

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + 4\sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Now applying the identity theorem ("comparison of coefficients") yields:
for $n = 0$: $2a_2 + 2a_0 = 0$, so $a_2 = -1$,
for $n = 1$: $6a_3 + 6a_1 = 0$, so $a_3 = 0$,
for $n \ge 2$: $(n^2+3n+2)a_{n+2} + (n^2+3n+2)a_n = 0$, so $a_{n+2} = -a_n$. So in general $a_{2n+1} = 0$, $a_{2n} = (-1)^n$. The power series has radius of convergence 1 and represents the function $x \mapsto (1+x^2)^{-1}$.