Solutions to final test Analysis 1 (2WA30)
April 2019

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- No rights can be derived from these solutions.
- Comparing your own solutions to the ones given here does not necessarily yield sufficient information on the former's correctness, as the problems often can be solved in various, different-looking ways.
- **Caution:** Reading these solutions can lead to an over-optimistic estimation of your abilities. It is no substitute to (trying to) solve the problems independently!

1. For any $k \in \mathbb{N}$, due to $V_k \subset V_{k+1} \subset V$ we have $s_k \leq s_{k+1} \leq \sup V$. Therefore the sequence (s_k) is increasing and bounded and therefore convergent. Moreover, as $s_k \leq \sup V$ for all k, we also have $\lim_{k\to\infty} s_k \leq \sup V$. .

To show the reverse inequality, fix $x \in V$. Then there is an $l \in \mathbb{N}$ such that $x \in V_l$. So

$$x \leq s_l \leq \sup\{s_k \mid k \in \mathbb{N}\} = \lim_{k\to\infty} s_k,$$

i.e. $\lim_{k\to\infty} s_k$ is an upper bound for V. $\sup V$ is the least upper bound, hence $\sup V \leq \lim_{k\to\infty} s_k$. .

2. For all $x \in \mathbb{R}$ we have $(x-1)^2 = x^2 - 2x + 1 \geq 0$ and thus $x \leq \frac{x^2+1}{2}$. Therefore both (a_n) and (b_n) are increasing. If they converge, the limit L has to satisfy $L = \frac{L^2+1}{2}$ which implies $L = 1$. However, we have $b_n \geq 2$ for all $n \in \mathbb{N}$, so (b_n) cannot converge to 1 and is therefore divergent . A simple induction argument shows that $a_n < 1$ for all $n \in \mathbb{N}$, so (a_n) converges, and the limit is 1.

3. If B is an accumulation point of (b_n), then there is a subsequence (b_{n_k}) with $b_{n_k} \to B$ as $k \to \infty$. The subsequence (a_{n_k}) is bounded because (a_n) is bounded. Hence, by Bolzano-Weierstrass, (a_{n_k}) has a convergent subsequence $(a_{n_{k_l}})$. Its limit is an accumulation point of (a_n), call it A_1. Now, because of

$$a_{n_{k_l}+1} = b_{n_{k_l}} - a_{n_{k_l}} \to B - A_1 \qquad \text{as } l \to \infty,$$

we see that $A_2 := B - A_1$ is also an accumulation point of (a_n).

4. **a)** As

$$\frac{\sqrt{\frac{n^2-2n+5}{n^3+6}}}{\sqrt{\frac{1}{n}}} \to 1 \qquad \text{as } n \to \infty$$

(Verify!) there exists an $n_0 \in \mathbb{N}$ such that

$$\sqrt{\frac{n^2-2n+5}{n^3+6}} \geq \frac{1}{2\sqrt{n}} \qquad \text{for } n \geq n_0.$$

So the (up to the factor 2) hyperharmonic series $\sum_{n=n_0}^{\infty} \frac{1}{2\sqrt{n}}$ is a divergent minorant for $\sum_{n=n_0}^{\infty} \sqrt{\frac{n^2-2n+5}{n^3+6}}$, and therefore the given series diverges.

b) Recall that due to $(a \pm b)^2 \geq 0$ for $a, b \in \mathbb{R}$ we have $\pm 2ab \leq a^2 + b^2$, hence also $|ab| \leq a^2 + b^2$. Consequently, the convergent series $\sum(a_n^2 + b_n^2)$ is a convergent majorant for $\sum |a_n b_n|$, and the series $\sum a_n b_n$ is absolutely convergent.

c) If $|x| < \min\{R_0, R_1\}$ then both $\sum_{k=0}^{\infty} a_{2k} x^{2k}$ and $\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$ converge, so also their sum $\sum_{k=0}^{\infty} a_k x^k$ does. If $|x| > \min\{R_0, R_1\}$, say $|x| > R_0$ without loss of generality, then there is an $\varepsilon > 0$ such that $\sqrt[2k]{|a_{2k}|} > (1+\varepsilon)/|x|$ for infinitely many indices k, and

$$|a_{2k} x^{2k}| \geq \left(\sqrt[2k]{|a_{2k}|}\,|x|\right)^{2k} \not\to 0,$$

so that $\sum_{k=0}^{\infty} a_k x^k$ is divergent. So the radius of convergence is $\min\{R_0, R_1\}$.

5. a)

$$\forall\, \varepsilon > 0 : \quad \exists\, x_0 \in \mathbb{R} : \quad \forall\, x \geq x_0 : \quad |f(x) - L| < \varepsilon, \tag{1}$$

$$\forall\, M > 0 : \quad \exists\, \delta > 0 : \quad \forall\, t \in (2-\delta, 2+\delta) \setminus \{2\} : \quad g(t) > M \tag{2}$$

b) As h is periodic, its range is $\{h(x) \,|\, x \in [0,T]\}$. The continuous function h takes on the bounded, closed interval $[0,T]$ its minimum value c in a point $x_1 \in [0,T]$ and its maximum value d in a point $x_2 \in [0,T]$. If $c < d$ then $x_1 \neq x_2$, say $x_1 < x_2$ without loss of generality, and by the Intermediate Value theorem applied to h on $[x_1, x_2]$, we find that all values in $[c,d]$ are taken by h on $[x_1, x_2]$.

c) We have to show

$$\forall\, \eta > 0 : \ \exists\, \mu > 0 : \ \forall\, t \in (2-\mu, 2+\mu) \setminus \{2\} : \quad |f(g(t) + h(t)) - L| < \eta.$$

Fix $\eta > 0$. Set $\varepsilon := \eta$ in (1). So

$$\exists\, x_0 \in \mathbb{R} : \ \forall\, x \geq x_0 : \quad |f(x) - L| < \eta. \tag{3}$$

Choose such an x_0 and set $M := x_0 - c$ in (2). So

$$\exists\, \delta > 0 : \ \forall\, t \in (2-\delta, 2+\delta) : \quad g(t) > x_0 - c. \tag{4}$$

Choose such a δ and set $\mu := \delta$.
Fix $t \in (2-\mu, 2+\mu) \setminus \{2\} = (2-\delta, 2+\delta) \setminus \{2\}$. Then by (4) and $h(t) \geq c$ we get $g(t) + h(t) \geq x_0$ and from this by (3) with $x = g(t) + h(t)$ we get $|f(g(t) + h(t)) - L| < \eta$.

6. a) The function $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by $F(z) = z - \sin z$ has derivative $F'(z) = 1 - \cos z \geq 0$ and is therefore increasing. As $F(0) = 0$ we have $F(z) \leq 0$ for $z \leq 0$ and $F(z) \geq 0$ for $z \geq 0$, This implies the statement.

b) Fix $x \in (-1,1)$. Because of $|\sin(x^n)| \leq |x^n| = |x|^n$, the convergent geometric series $\sum |x|^n$ is a convergent majorant for $\sum |\sin(x^n)|$, and therefore $\sum \sin(x^n)$ converges absolutely.

c) Similarly, for $x \in [-a, a]$ we have $|\sin(x^n)| \leq a^n$, the sum $\sum a^n$ is convergent, and the function series is uniformly convergent by the Weierstrass criterion.

d) Fix $x \in (-1,1)$ and choose $a \in (|x|, 1)$. The terms $x \mapsto \sin(x^n)$ are continuous, and due to the uniform convergence, the sum function s is continuous on $[-a, a]$, so in particular in x.

e) Let $f_n : (-1,1) \longrightarrow \mathbb{R}$ be given by $f_n(x) = \sin(x^n)$. Then all f_n are continuously differentiable, with derivatives given by $f_n'(x) = nx^{n-1}\cos(x^n)$. For $a \in (0,1)$ the series $\sum f_n'$ converges uniformly on $[-a,a]$ by the Weierstrass criterion, as for $x \in [-a,a]$ we have $|f_n'(x)| \le na^{n-1}$, and the series $\sum na^{n-1}$ is convergent. So s is differentiable on $[-a,a]$ for all $a \in (0,1)$, and therefore as well on $(-1,1)$.

7. The additional conditions imply $a_0 = 1$, $a_2 = 0$. Inserting the ansatz in the equation yields

$$(x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^{n-1} + 4x^3\sum_{n=0}^{\infty} a_n x^n = 0$$

and after rearranging and shifting the index

$$\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 4\sum_{n=3}^{\infty} a_{n-3}x^n = 0.$$

Now applying the identity theorem (“comparison of coefficients”) yields:
for $n = 0$: $a_1 = 0$,
for $n = 2$: $6a_3 - 3a_3$, so $a_3 = 0$,
for $n \ge 3$: $(n+1)(n-1)a_{n+1} = -4a_{n-3}$.

So in general $a_{4n+1} = a_{4n+2} = a_{4n+3} = 0$, $a_{4n} = (-1)^n/(2n)!$, $n \in \mathbb{N}$. The power series converges on all of $\mathbb{R}$ and represents the function $x \mapsto \cos(x^2)$.