Solutions intermediate test Analysis 1 december 2018

No rights can be derived from these solutions.

1. As
$$\frac{|a|}{|a|+1} < 1 \quad \text{for all } a \in A,$$
the number 1 is an upper bound for B, and therefore $\sup B \leq 1$. On the other hand, suppose $1-\varepsilon$ would be an upper bound for some $\varepsilon > 0$. Then
$$\frac{|a|}{|a|+1} \leq 1-\varepsilon \quad \text{and therefore} \quad |a| \leq \frac{1}{\varepsilon} - 1 \quad \text{for all } a \in A,$$
which contradicts the unboundedness of A. So B has no bounds smaller than 1, and therefore $\sup B = 1$.

Alternative: $\sup B \leq 1$ as above. Suppose $\sup B < 1$. Then
$$\frac{|a|}{|a|+1} \leq \sup B \quad \text{and therefore} \quad |a| \leq \frac{\sup B}{1-\sup B} \quad \text{for all } a \in A,$$
which contradicts the unboundedness of A. So $\sup B = 1$.

2. From
$$\frac{1}{n+2018} \leq c_n \leq \frac{1}{n-1}$$
we have
$$\left(1+\frac{1}{n+2018}\right)^n \leq (1+c_n)^n \leq \left(1+\frac{1}{n-1}\right)^n \tag{1}$$
for $n \geq 2$. By standard limits and limit theorems,
$$\left(1+\frac{1}{n+2018}\right)^n = \underbrace{\left(1+\frac{1}{n+2018}\right)^{n+2018}}_{\to e} \underbrace{\left(1+\frac{1}{n+2018}\right)^{-2018}}_{\to 1} \to e,$$
as well as
$$\left(1+\frac{1}{n-1}\right)^n = \underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{\to e} \underbrace{\left(1+\frac{1}{n-1}\right)} \to 1 \to e.$$
So by the squeeze theorem, $(1+c_n)^n \to e$ as well.

3. As
$$\frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} \to 1 \qquad \text{as } n \to \infty,$$
we have that there is an $n_0 \in \mathbb{N}$ such that
$$\frac{n^2}{n^4-1} \leq \frac{2}{n^2} \qquad \text{for all } n \geq n_0.$$

Consequently,

$$\sum_{n=n_0}^{\infty} \frac{2}{n^2} \quad \text{(hyperharmonic series) is a convergent majorant for} \quad \sum_{n=n_0}^{\infty} \frac{n^2}{n^4-1},$$

and therefore the series $\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$ is convergent as well.

4. Observe first that because of $a_n \to 0$, for all $k \in \mathbb{N}_+$ there is an $m_k \in \mathbb{N}_+$ such that

$$a_n < \frac{1}{k^2} \quad \text{for all } n \geq m_k.$$

Let $n_1 = m_1$ and for $k \geq 2$ define $n_k := \max\{m_k, n_{k-1}+1\}$. This ensures that (n_k) is an index sequence with $a_{n_k} < \frac{1}{k^2}$. Therefore, the series

$$\sum_k a_{n_k} \quad \text{has the convergent majorant} \quad \sum_k \frac{1}{k^2}$$

and is therefore convergent.