Solutions intermediate test Analysis 1 december 2019

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1. **a)** The statement is true. For every $\varepsilon > 0$ there is an $a \in A$ and a $b \in B$ such that $a > \sup A - \varepsilon$ and $b < \inf B + \varepsilon$. Choose $\varepsilon := (\sup A - \inf B)/2$ which is positive by assumption. So there are $a \in A$ and $b \in B$ such that

$$a > \sup A - \varepsilon = (\sup A + \inf B)/2 = \inf B + \varepsilon > b.$$

b) The statement is false. For a counterexample, choose $A = \{1, 4\}$, $B = \{2, 3\}$.

2. **a)** The statement is true. We have

$$\forall\, \varepsilon > 0 : \quad \exists\, n_0 \in \mathbb{N} : \quad \forall\, n \geq n_0 : \quad x_n - 1 < \varepsilon. \tag{1}$$

We need to show:

$$\forall\, M > 0 : \quad \exists\, n_0 \in \mathbb{N} : \quad \forall\, n \geq n_0 : \quad \frac{x_n}{x_n - 1} > M. \tag{2}$$

Fix $M > 0$ and choose $\varepsilon := 1/M$ (1). So

$$\exists\, n_0 \in \mathbb{N} : \quad \forall\, n \geq n_0 : \quad x_n - 1 < 1/M.$$

Choose such an n_0 and fix $n \geq n_0$. Then $\frac{x_n}{x_n-1} > M$. This proves (2).

b) The statement is true. For n sufficiently large we have $1 \leq x_n \leq 2 \leq n$, so for these n it follows that

$$1 \leq \sqrt[n]{x_n} \leq \sqrt[n]{n},$$

and the result follows by the standard limit $\sqrt[n]{n} \to 1$ and the squeeze theorem.

c) The statement is false. For a counterexample, choose $x_n = 1 + 1/n$. Then $x_n^n \to e \neq 1$.

3. **a)** (Idea: The terms behave as $(1/2)^k$ for large k. Therefore the series converges.)
As $\frac{k+1}{2k-1} \to \frac{1}{2}$ for $k \to \infty$, we have that there is a k_0 such that $\frac{k+1}{2k-1} < 2/3$ for all $k \geq k_0$. Therefore, the geometric series $\sum_{k=k_0}^{\infty} \left(\frac{2}{3}\right)^k$ is a convergent majorant for $\sum_{k=k_0}^{\infty} \frac{k+1}{2k-1}$, so also $\sum_{k=1}^{\infty} \frac{k+1}{2k-1}$ converges.
Alternatively, the result follows directly from the root test.

b) (Idea: The terms behave as $1/k$ for large k. Therefore the series diverges.)
Let M be an upper bound for (a_k). Then, for $k > \sqrt{M}$,

$$\frac{k + a_k}{k^2 + a_{k+1}} > \frac{k}{k^2 + M} > \frac{1}{2k}.$$

So with $k_0 > \sqrt{M}$, the harmonic series $\sum_{k=k_0}^{\infty} \frac{1}{2k}$ is a divergent minorant for $\sum_{k=k_0}^{\infty} \frac{k+a_k}{k^2+a_{k+1}}$, so also $\sum_{k=1}^{\infty} \frac{k+a_k}{k^2+a_{k+1}}$ diverges.