Exam on “Non-linear Optimization (2DME20)” TU/e
Test exam, Fall 2015

There are 8 questions worth a total of 80 points. You are **not allowed** *to use any tools other than pen and paper: no books, no notes, no pocket calculators!*

(1) (a) Formulate Jensen's inequality for a convex function $f : \mathbb{R} \to \mathbb{R}$ and $n = 3$ variables.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Prove that the minimization of $f(a) + f(b) + f(c)$ subject to the constraint $a + b + c = 3$ has $3f(1)$ as optimal objective value.
(c) Determine the minimum value of $1/\sin\alpha + 1/\sin\beta + 1/\sin\gamma$ where α, β, γ are the angles of a triangle in the plane.

(2) Prove that $f(X) = \log\det X$ is a concave function on the set S^n_{++} of positive definite matrices.

(3) For matrices $P, Q \in S^n$, consider the problem $\min\{x^T P x \mid x^T Q x \geq 1,\ x^T x = 1\}$.
(a) Is this a convex optimization problem?
(b) Give the Lagrangian, the Lagrange dual function, and the Lagrange dual.

(4) An instance of THREE-CYCLE consists of an undirected graph $G = (V, E)$ with $|V| = 3n$. The question is whether the vertices of G can be covered by three simple cycles of length n.
(a) Prove that THREE-CYCLE lies in NP.
(b) Prove that THREE-CYCLE is NP-hard.

(5) (a) State the ILP for unit communication delay scheduling.
(b) Prove that the integrality gap of the LP relaxation is at least 4/3.

(6) Let $A_0, A_1, \ldots, A_k$, B, C be symmetric $n \times n$ matrices and let $q \in \mathbb{R}$. We want to find a matrix A in the set $\{A_0 + x_1 A_1 + \cdots + x_k A_k \mid x \in \mathbb{R}^k\}$ such that no eigenvalue of BA exceeds the bound q and such that the smallest eigenvalue of CA is as large as possible. Formulate this problem as a semi-definite optimization problem.

(7) Are the three functions in (a)–(c) self-concordant? Prove.
(a) $f(x) = x^6/30$ on $\mathbb{R}$
(b) $f(x) = x\log x - 8\log x$ on $\mathbb{R}_{++}$
(c) $f(x) = -\log(x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2)$ on $\mathbb{R}^4$
(d) Prove or disprove: If f is self-concordant on $\mathbb{R}$, then $\frac{1}{2}f$ is self-concordant on $\mathbb{R}$.

(8) (a) Define when a search direction Δx is a *descent direction.*
(b) Show that for strictly convex f, the Newton direction Δx_{nt} is a descent direction. (Recall that $\Delta x_{nt} = -(\nabla^2 f(x))^{-1}\nabla f(x)$.)
(c) Define the *degree* of a generalized logarithm for a cone K.
(d) State (without proof) a generalized logarithm for the semi-definite cone.

Point distribution: (1) 3,3,4 (2) 10 (3) 2,8 (4) 2,8 (5) 5,5 (6) 10 (7) 2,2,2,4 (8) 2,3,3,2