Optimization (2MMD10/2DME20) TU/e, fall 2015
Problem set for weeks 1,2,3

Exercise 1. Some fundamental geometric optimization problems.

(a) Among all triangles of perimeter 12, find the one that has maximal area.

(b) Among all rectangles of perimeter 12, find the one that has maximal area.

(c) Determine the Steiner point of the rectangle $ABCD$.

Exercise 2. Determine the Steiner point for $n \geq 3$ points $P_1, P_2, \ldots, P_n$ lying in that order on a straight line. (Hint: Start with the cases $n = 3$ and $n = 4$.)

Exercise 3. Given a closed straight line segment, how should we choose $n \geq 2$ (not necessarily distinct) points on the segment so that the sum of all distances between pairs of points is maximized?

Exercise 4. Consider a wedge-shaped region between two lines ℓ_1 and ℓ_2, and let M be a given point in the interior of this region. Determine a line ℓ through M that cuts off (from the wedge-shaped region) a triangle of minimal area.

(Hint: Let Q be the corner of the wedge, that is, the intersection point of ℓ_1 and ℓ_2. Let P be a point in the wedge, so that M is the mid-point of PQ. Construct a parallelogram with center in M, a diagonal PQ, and sides parallel to ℓ_1 and ℓ_2. The solution is given by a diagonal of this parallelogram.)

Exercise 5. Let x, y, z be positive real numbers. True or false?

(a) $x + y > z$ implies $x^2 + y^2 > z^2$

(b) $x + y > z$ implies $\sqrt{x} + \sqrt{y} > \sqrt{z}$

(c) $|\max\{x, y, z\}| \;=\; \max\{|x|, |y|, |z|\}$

(d) $(\max\{x, y, z\})^3 \;=\; \max\{x^3, y^3, z^3\}$

Exercise 6. Prove that all non-negative real numbers x satisfy the two inequalities $x + \frac{1}{x} \geq 2$ and $x^3 + 1 \geq x(x + 1)$. When does equality hold?

Exercise 7. For real numbers x, y, z, prove that $x^2 + y^2 + z^2 \geq xy + xz + yz$. When does equality hold?

Exercise 8. The *arithmetic-geometric mean inequality* for $n \geq 2$ non-negative real numbers $x_1, \ldots, x_n$ states that

$$(x_1 x_2 \cdots x_n)^{1/n} \;\leq\; \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The left hand side of this inequality is called the *geometric mean* of the numbers $x_1, \ldots, x_n$ and the right hand side is called their *arithmetic mean.*

(a) Verify that the inequality holds in case $x_1 = x_2 = \cdots = x_n$.

(b) Prove the inequality for $n = 2$. (Hint: rewrite the inequality so its right hand side becomes a square and its left hand side becomes zero.)

(c) Prove that the correctness of the inequality for n implies its correctness for $2n$. (Hint: Split the $2n$ numbers into two groups $x_1, \ldots, x_n$ and $x_{n+1}, \ldots, x_{2n}$; apply the inequality for n to each group in the left hand side; then apply the inequality for $n = 2$.)

(d) Prove that the correctness of the inequality for n implies its correctness for $n - 1$.

(e) Argue that (b), (c), (d) together imply the correctness of the inequality for all $n \geq 2$.

Exercise 9. Find the largest value of xyz if x, y, z are positive real numbers satisfying $2x + 3y + 4z = 36$. (Hint: Use the arithmetic-geometric mean inequality.)

Exercise 10. Prove the *arithmetic-harmonic mean inequality* for $n \geq 2$ non-negative real numbers $x_1, \ldots, x_n$, which states that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

The right hand side of this inequality is called the *harmonic mean* of the numbers $x_1, \ldots, x_n$. (Hint: Use the arithmetic-geometric mean inequality to show that the geometric mean is sandwiched between the arithmetic and the harmonic mean.)

Exercise 11. Prove the *Cauchy inequality* for real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, which states that

$$(a_1^2 + a_2^2 + \cdots + a_n^2)\,(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

(Hint: The case where all b_i are zero is trivial. Otherwise consider the sum $(a_1 + xb_1)^2 + \cdots + (a_n + xb_n)^2$ for an arbitrary real number x. This sum yields a quadratic polynomial in x. As the quadratic polynomial is non-negative, its discriminant is non-positive.)

When does equality hold?

Exercise 12. Deduce the following inequalities from the Cauchy inequality.

(a) The arithmetic-harmonic mean inequality.

(b) Real numbers $a_1, \ldots, a_n$ and positive real numbers $b_1, \ldots, b_n$ always satisfy

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

(c) Positive real numbers $x_1, \ldots, x_n$ satisfy the arithmetic-quadratic mean inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 \cdots + x_n^2}{n}}.$$

The right hand side of this inequality is called the *quadratic mean* of the numbers $x_1, \ldots, x_n$.

Exercise 13. Determine the minimum of $f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{5}$ where $x, y, z \geq 0$ are real numbers with $x + y + z = 10$. (Hint: Use the preceding exercise.)

Exercise 14. Of all rectangular parallelepipeds inscribed in a sphere of radius 1, find the one of largest volume.

Exercise 15. A group of n remote communities wants to build a central warehouse. The location of the ith community is given by coordinates (x_i, y_i) in the plane. Goods will be delivered by plane from the warehouse to the communities, and the ith community needs d_i deliveries per month. The goal is to locate the warehouse so that the total travel distance of all deliveries is minimized. Formulate this problem as an NLP.

Exercise 16. A chimney needs cleaning regularly, as soot builds up on its inside at a rate of 1 cm per month. The cost of cleaning when the soot has thickness x cm amounts to $13+x^2$ Euro. Cleaning can occur at the end of any month. The chimney is clean at the start of month 1, and it must be cleaned exactly nine times in the coming 48 months. At least three cleanings must occur in the first 15 months, and the final cleaning must be at the end of month 48.

Formulate a non-linear program to determine in which months the cleaning should occur, so that the total cost is minimized. All constraints should be either linear constraints or integrality constraints.

Exercise 17. Given an undirected graph with positive real edge weights. How do you find a spanning tree (a) that has maximum weight; (b) that minimizes the difference between longest and shortest edge; (c) that minimizes the product of all edge lengths.

Exercise 18. Give an LP-formulation of the max-flow problem.

Exercise 19. Consider the following problem: given an undirected graph $G = (V, E)$ and two vertices $s, t \in V$, decide whether there exists a simple path from s to t that uses an even number of edges.

Formulate this even-path problem as a special matching problem. (Hint: Make a copy G' of G. Take the union of G and G', remove s' and t from it, and add an edge between every vertex $v \in V - \{s, t\}$ and its copy v' in G'.)

Exercise 20. A vertex cover C in a graph $G = (V, E)$ is a subset of the vertices, so that every edge in E has at least one end-vertex in C.

Give an ILP formulation for the problem of finding a vertex cover of smallest possible size.

Exercise 21. Give an ILP formulation for the problem of placing n queens on an $n \times n$ chessboard, so that no two queens share any row, column or diagonal.

Exercise 22. Let $G = (V, E)$ be an undirected graph with vertex set $V = \{1, 2, \ldots, n\}$. Consider the following continuous optimization problem.

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i^2 + 2 \cdot \sum_{[i,j] \notin E} x_i x_j \\ s.t. \quad & \sum_{i=1}^{n} x_i = 1 \\ & 0 \le x_i \le 1 \qquad \text{for } 1 \le i \le n. \end{aligned}$$

(a) Show that there exists an optimal solution $x_1^*, \ldots, x_n^*$ for which $[i,j] \notin E$ implies $x_i^* x_j^* = 0$.
(b) Show that the support $P = \{i | x_i^* > 0\}$ of this optimal solution forms a clique of maximum size in G.

Exercise 23. A 0-1 program is an integer optimization problem where every variable is a binary variable (that is, it can only take the values 0 or 1).

Show that every 0-1 program can be expressed as a continuous non-linear program. (Hint: use equations with left hand side $x_i(x_i - 1)$.)

Exercise 24. Consider a non-linear integer program (P) where the objective function and all constraints are given as polynomial functions. Show that (P) can be rewritten into an equivalent non-linear integer program (Q) where the objective function and all constraints are given as polynomial functions of degree 2.

Exercise 25. Are the following sets convex?
(a) $\{(x,y) \in \mathbb{R}^2 \mid x + 2y \leq 10,\ 5x + y \leq 4\}$
(b) $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 4 \text{ or } (x+1)^2 + y^2 \leq 4\}$
(c) $\{x \in \mathbb{R} \mid \cos(x) \leq 0\}$
(d) $\{x \in \mathbb{R}^3 \mid \|x\| \geq 1\}$
(e) $\{\begin{bmatrix} 3x+y \\ -y-2 \end{bmatrix} \mid x \geq 0,\ y \leq 0,\ x^2 + y^2 \leq 1\}$
(f) $\{x \in \mathbb{R} \mid x^3 - 2x^2 + x \geq 0\}$
(g) $\{x \in \mathbb{R}^3 \mid x_1 + x_2 e^t + x_3 e^{2t} \geq 2 \text{ for all } t \leq 3\}$
(h) The set of real 2×2 matrices A with $\det(A) \geq 1$

Exercise 26. A cake is to be divided among $n \geq 2$ children so that the ith child receives a fraction x_i of the cake. The vector $x = (x_1, \ldots, x_n)$ is called an *allocation*. An allocation is *feasible*, if it satisfies the following three porperties: (i) Every child must receive some non-zero share of the cake. (ii) The entire cake must be allocated to the children. (iii) The first child $(i = 1)$ must be allocated a share that is at least twice as big as the share of any other child.

Show that the set S of all feasible allocations is convex.

Exercise 27. Are the following quadratic functions convex?
(a) $f(x) = x_1^2 + x_2^2 + 2x_1x_2 + 5x_1 - x_2 + 1$ on $\mathbb{R}^2$
(b) $f(x) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$ on $\mathbb{R}^3$

Exercise 28. Are the following functions convex on the given domain? Are they concave?
(a) $f(x) = \arctan(x)$ on $\mathbb{R}^+$
(b) $f(x,y,z) = (y+2z)^2/(x-3y)$ on $\{(x,y,z) \in \mathbb{R}^3 \mid x - 3y > 0\}$
(c) $f(x,y,z) = x^2 + y^2 + 5z^2 - xy - xz - 3yz$ on $\mathbb{R}^3$
(d) $f(x) = \max\{\|x\|, \|x-a\|\}$ on $\mathbb{R}^n$, where $a \in \mathbb{R}^n$
(e) $f(x,y) = -\log(\exp(x) + \exp(y))$ on $\mathbb{R}^2$
(f) $f(u,x,y,z) = u \log u + x \log x + y \log y + z \log z$ on $\mathbb{R}^4_+$

Exercise 29. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and let function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = (f(x) - 2)^2$.
(a) Give an example of f, so that g is not convex.
(b) Prove: If $f(x) \geq 2$ for all $x \in \mathbb{R}$, then g is convex.

Exercise 30. Prove or disprove:
(a) The square of a convex positive function is convex.
(b) The square of a concave positive function is concave.
(c) The reciprocal of a positive concave function is convex.

Exercise 31. Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is affine (that is, $f : x \mapsto cx + d$ for some $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$), if and only if f is both convex and concave.

Exercise 32. Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = ||x||^2$ is strictly convex.

Exercise 33. Prove that a strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$ has at most one minimizer.

Exercise 34. Consider the quadratic function $f(x) = \frac{1}{2}x^T Ax + cx$, where A is a symmetric $n \times n$ matrix. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded from below.
(a) Show that A is positive semi-definite.
(b) Show that f attains its minimum.

Exercise 35. Determine the convex hull of the set $S_1 \cup S_2$ where $S_1 = \{(0, 0, 0)\}$ and $S_2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1;\ x_3 = 1\}$.

Exercise 36. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Prove that for $x < y < z$, we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}$$

Exercise 37. Prove for all positive real numbers x and y that

$$\frac{x}{3} + \frac{2y}{3} \ \leq\ \sqrt{\ln\left(\frac{1}{3}e^{x^2} + \frac{2}{3}e^{y^2}\right)}.$$

(Hint: Use the convexity of an appropriately chosen function.)

Exercise 38. Prove for all real numbers a, b, c, d that

$$\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{12} + \frac{d}{12}\right)^4 \ \leq\ \frac{1}{2}a^4 + \frac{1}{3}b^4 + \frac{1}{12}c^4 + \frac{1}{12}d^4.$$

Exercise 39. Use Jensen's inequality to determine the maximum value of $\sin\alpha + \sin\beta + \sin\gamma$ where α, β, γ are the angles of a triangle.

Exercise 40. Use the convexity of $f(z) = z + \frac{1}{z}$ for positive real z to determine the minimum value of

$$g(x, y) \ =\ 2x^2 + 3y^2 + \frac{1}{2x^2 + 3y^2}$$

where x, y are positive real numbers.

Exercise 41. Prove Young's inequality, which states for real numbers $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for positive real numbers x and y that

$$xy \;\leq\; \frac{x^p}{p} + \frac{y^q}{q}.$$

(Hint: Use the arithmetic-geometric mean inequality.) When does equality hold?

Exercise 42. Show that the master inequality implies

(a) the concave version of Jensen's inequality;

(b) the Hölder inequality, which states for two real numbers $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ that

$$\sum_{i=1}^{n} a_i b_i \;\leq\; \left(\sum_{i=1}^{n} {a_i}^p\right)^{1/p} \left(\sum_{i=1}^{n} {b_i}^q\right)^{1/q}$$

(c) the Minkowski inequality, which states for a real parameter $p > 1$ and for positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ that

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p} \;\leq\; \left(\sum_{i=1}^{n} {a_i}^p\right)^{1/p} + \left(\sum_{i=1}^{n} {b_i}^p\right)^{1/p}.$$

(d) the Milne inequality, which states for positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$

$$\left(\sum_{i=1}^{n} (a_i + b_i)\right) \left(\sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}\right) \;\leq\; \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right)$$

Exercise 43. Let $U \subset \mathbb{R}^n$ be a compact convex non-empty set, and let $v \in \mathbb{R}^n$.
(a) Prove that there exists a unique point in U that is closest to v.
(b) Give an example so that exactly two points in U are farthest from v.
(c) Give an example so that infinitely many points in U are farthest from v.

Exercise 44. Let $U \subseteq \mathbb{R}^n$ be a convex set, and let $f : U \to \mathbb{R}$ be a convex function.
(a) Prove that every local minimum of f is a global minimum.
(b) Prove that for strictly convex f, every local maximum of f is an extreme point of U.

Exercise 45. Show that for real $u, x, y, z > 0$, the following function has no minimum:

$$f(u, x, y, z) \;=\; \frac{1}{x^3} + \frac{2z^4}{y^6} + 3u^3 z^2 + \frac{5x^2 y^4 z^2}{u}$$

Exercise 46. Let q be real parameter. Use the KKT conditions to find all stationary points of the function $f(x, y) = x^3 - 3qxy + y^3$. Determine all local minima, local maxima, and saddle points.

Exercise 47. Find all real values q, for which the function $f(x, y) = x^2 - qxy^2 + 2y^4$ has two distinct stationary points.

Exercise 48. Use the KKT conditions to find all stationary points of the following functions. Determine all local minima, local maxima, and saddle points.
(a) $f(x, y) = \exp(x^2 + y^2) - x^2 - 2y^2$
(b) $f(x, y, z) = xy \exp(-x - y - z)$
(c) $f(x, y, z) = 2x^2 + y^2 + z^2 + xy + yz - 6x - 7y - 8z$
(d) $f(x, y) = (x^2 - 1)^2 + (xy - x - 1)^2$

Exercise 49. Is the following NLP convex? Determine all stationary points. Determine an optimal solution.

$$\begin{array}{ll} \min & x_1^2 - x_2^2 \\ \text{s.t.} & x_2 - 1 \quad \leq 0 \\ & -x_2 \qquad \leq 0 \end{array}$$

Exercise 50. Write down the KKT optimality conditions for the point $x^* = (1, 1)$ in the following NLP. Is x^* a global optimum?

$$\begin{array}{lll} \min & -7x_1 - 5x_2 & \\ \text{s.t.} & 2x_1^2 + x_2^2 + x_1x_2 - 4 & \leq 0 \\ & x_1^2 + x_2^2 - 2 & \leq 0 \\ & -2x_1 + 1 & \leq 0 \end{array}$$

Exercise 51. Write down the KKT optimality conditions for the point $x^* = (1/2, -1/2, 1/2)$ in the following NLP. Is x^* a global optimum?

$$\begin{array}{lll} \min & x_3 & \\ \text{s.t.} & x_1 + x_2 & \leq 0 \\ & x_1^2 - 4 & \leq 0 \\ & x_1^2 - 2x_1 + x_2^2 - x_3 + 1 & \leq 0 \end{array}$$

Exercise 52. Write down the KKT optimality conditions for the point $x^* = (0, 1/2)$ in the following NLP. Is x^* a global optimum?

$$\begin{array}{lll} \min & 4x_1^2 + 2x_2^2 - 6x_1x_2 + x_1 & \\ \text{s.t.} & 2x_1 - x_2 & \leq 0 \\ & 2x_1 - 2x_2 & \leq -1/2 \\ & x_1 \leq 0, \ x_2 \geq 0 & \end{array}$$

Exercise 53. Write down the KKT optimality conditions for the point $x^* = (1, 0)$ in the following NLP. Is x^* a global optimum?

$$\begin{array}{lll} \min & x_1^2 + 3x_2^2 - x_1 & \\ \text{s.t.} & x_1^2 - x_2 & \leq 1 \\ & x_1 + x_2 & \geq 1 \end{array}$$

Exercise 54. Find the minimum value of the function $f(x, y) = (x - 2)^2 + (y - 1)^2$ subject to the conditions $y \geq x^2$ and $x + y \leq 2$.

Exercise 55. Let $q > 0$ be a real parameter. Write down the KKT conditions for the point $x^* = (0, 0)$ in the following NLP. For which values of q is x^* a global optimum?

$$\begin{aligned} \min \quad & x^2 + (y-1)^2 \\ \text{s.t.} \quad & qx^2 - y \geq 0 \end{aligned}$$

Exercise 56. Prove that the following NLP is convex. Use the KKT conditions to find an optimal solution.

$$\begin{aligned} \min \quad & 2x_1 \arctan(x_1) - \ln(x_1^2 + 1) + x_2^4 + (x_3 - 1)^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 - 4 \leq 0 \\ & -x_1 \leq 0 \end{aligned}$$

Exercise 57. For a real parameter q, use the KKT conditions to find an optimal solution to the following NLP.

$$\begin{aligned} \max \quad & x + qy \\ \text{s.t.} \quad & x^2 + y^2 \leq 1 \\ & x + y \geq 0 \end{aligned}$$

Exercise 58. Argue that the KKT conditions are necessary and sufficient conditions for the optimal solution to the following NLP. Solve the NLP.

$$\begin{aligned} \min \quad & x_1^2 + x_1 + x_2 + x_2^2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq -1 \\ & 2x_1 + x_2 \leq 2 \end{aligned}$$

Exercise 59. Consider the following NLP.

$$\begin{aligned} \min \quad & x_3 \\ \text{s.t.} \quad & (x_1 - 1)^2 + x_2^2 + x_3^2 - 1 \geq 0 \\ & (x_1 + 1)^2 + x_2^2 + x_3^2 - 1 \geq 0 \\ & x_1^2 + x_2^2 + x_3^2 - 4 \leq 0 \end{aligned}$$

(a) Prove that the feasible region U is not convex.
(b) Compute the convex hull of U, and express this convex hull as the intersection of a halfspace and another simple convex set.
(c) Find the optimal solutions minimizing x_3 over the convex hull of U. Which of these solutions are optimal solutions for the original problem?

Exercise 60. Determine all stationary points and all global maximizers for the following NLP.

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^5 x_i^3 \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^5 x_i^2 = 1 \end{aligned}$$

Exercise 61. For real numbers $c_1, \dots, c_n$ with $c_1 < 0$ and $c_2, \dots, c_n > 0$, consider the following NLP and determine an optimal solution.

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^n c_i x_i \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^n x_i^2 \leq 1 \\ & x_1, \dots, x_n \geq 0 \end{aligned}$$

Exercise 62. For real numbers $c_1, \ldots, c_n > 0$ and $b > 0$, find an optimal solution for the following NLP and show that it is unique.

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i x_i^2 \\ \text{s.t.} & \sum_{i=1}^n x_i \ = b \end{array}$$

Exercise 63. Formulate the following problems as NLPs, determine the KKT conditions, and find the optimal solutions.

(a) Find the area of the largest isosceles triangle ABC (with $|AC| = |BC|$) that is contained in the unit circle.

(b) Find the maximum surface area of a rectangular box whose twelve edges have total length 24.

(c) Find a point P in the plane that minimizes the sum of the squared distances from three given points A, B, C.

(d) Find a point P in the plane that minimizes the sum of the distances from three given points A, B, C. (You may assume that all angles of the triangle ABC are smaller than $2\pi/3$.)

Exercise 64. Show that every convex program (P) can be rewritten into an equivalent convex program (Q) with linear objective function.

Exercise 65. Determine the minimum of

$$f(x, y, z) \ = \ \frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2}$$

subject to the constraints $xyz = 1$ and $x \neq 1$, $y \neq 1$, $z \neq 1$.

Exercise 66. Give the Lagrangian, Lagrange dual function, and Lagrange dual of the problem $\min\{x^T x \mid Ax = b, x \geq 0\}$

Exercise 67. Give the Lagrangian, Lagrange dual function, and Lagrange dual of the problem $\min\{\sum_{i=1}^3 x_i \log(x_i)\} \mid x_1 + x_2 + x_3 = 1,\ x_1 + 2x_2 + 4x_3 \leq 2\}$

Exercise 68. Give the Lagrangian, Lagrange dual function, and Lagrange dual of the problem $\min\{x^T P x \mid x^T Q x \geq 1,\ x^T x = 1\}$