Optimization (2MMD10/2DME20), lecture 1a

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Our program for week 1

Tuesday:

- lots of examples from continuous optimization
  (mainly very old, mainly Greek, mainly geometric)

Friday:

- lots of examples from discrete optimization
  (mainly from the 20th century)
- some impossibility results

What is optimization about?

Definition of a generic optimization problem

Given:

a set S of mathematical objects;

a cost function $c : S \to \mathbb{R}$;

Goal:

find an object $s^* \in S$ so that $c(S^*)$ is minimal

- minimization: minimize the cost
- maximization: maximize the profit
- $\max c(s) \iff -(\min -c(s))$
- continuous: S is subset of $\mathbb{R}^n$
- discrete: S is finite, or subset of $\mathbb{Z}^n$
- (stochastic: S and/or c are stochastic)

Heron's Problem (1)

- Heron of Alexandria (10–75 AC)
  formula for the area of a triangle: $A = \sqrt{s(s-a)(s-b)(s-c)}$

Problem

Given two points A and B on one side of a straight line ℓ,
find point C on line such that $|AC| + |CB|$ is as small as possible.

- Euclid of Alexandria ($\sim$ 300 BC): father of geometry

Equal Angle Law of Reflection (or Euclid's Law of Reflection)

If a beam of light is sent toward a mirror, then the angle of incidence equals the angle of reflection.

Heron's Problem (2)

- Willebrord Snel van Royen = Willebrord Snellius (1580–1626)
  Dutch astronomer and mathematician

Snell's law of refraction

(A formula used to describe the relationship between the angles of incidence and refraction, when a light ray waves passes through the boundary between two different isotropic media.)

The ratio of the sine of the incidence angle to the sine of the reflection angle is a constant that is independent of the incidence angle.

- Christiaan Huygens (1629–1695): proof by geometric arguments
- Leibniz: proof using derivatives (1684)

Fermat's principle:

In an inhomogeneous medium, light travels from one point to another along the path requiring the shortest time.

Dido's Problem (1)

Dido:

- founder and first queen of Carthage (in modern-day Tunisia)
- primarily known from the "Aeneid" by the Roman poet Virgil

> The Kingdom you see is Carthage, the Tyrians, the town of Agenor;
> But the country around is Libya, no folk to meet in war.
> Dido, who left the city of Tyre to escape her brother,
> Rules here – a long and labyrinthine tale of wrong
> Is hers, but I will touch on its salient points in order...
> Dido, in great disquiet, organised her friends for escape.
> They met together, all those who harshly hated the tyrant
> Or keenly feared him: they seized some ships which chanced to be ready...
> They came to this spot, where to-day you can behold the mighty
> Battlements and the rising citadel of New Carthage,
> And purchased a site, which was named 'Bull's Hide' after the bargain
> By which they should get as much land as they could enclose with a bull's hide.

Dido's Problem (2)

A certain spot on the coast of what is now the bay of Tunis caught Dido's fancy, and she negotiated the sale of land with the local leader, Yarb. She asked for very little: as much as could be *encircled with a bull's hide*. Dido managed to persuade Yarb, and a deal was struck.
Dido then cut a bull's hide into narrow strips, tied them together, and enclosed a large tract of land. On this land she built a fortress and, near it, the city of Carthage.

Dido's problem

Among all closed plane figures of a given perimeter P,
find the one that encloses the largest area.

Quote (Pythagoras)

The most beautiful solid is the sphere,
and the most beautiful plane figure the circle.

Dido's Problem (3)

- Zenodorus (200 BC – 140 BC)

Theorem (Zenodorus)

Consider a polygon of perimeter P with n sides and maximum area.
1. The polygon must be convex.
2. All sides of the polygon must have equal length.
3. All angles of the polygon must have equal size.

Lemma (modern language)

For a regular n-gon inscribed in a circle of radius r,
the perimeter is $P = 2nr\sin(\pi/n)$,
the area is $A = r\cos(\pi/n)P/2$.

Hence: $P^2 \; = \; 4nA\tan(\pi/n) \; \geq \; 4\pi A$.
(which follows from $\tan x \geq x$ for $0 \leq x \leq \pi/2$)

Dido's Problem (4)

Lemma (modern language)

For every perimeter P and every $n \geq 3$,
there exists a maximum area polygon of perimeter P with n sides.

Lemma (modern language)

For every closed plane curve of perimeter P^* that encloses an area A^*,
and for every real $\varepsilon > 0$,
there exists a polygon of perimeter P and area A such that
$|P - P^*| \leq \varepsilon$ and $|A - A^*| \leq \varepsilon$

Theorem (modern language)

The area enclosed by an arbitrary closed curve of length P
does not exceed the area enclosed by a circle of perimeter P.

Euclid's Problem (1)

- Euclid of Alexandria ($\sim$ 300 BC): father of geometry
- Euclid's "Elements": first scientific monograph and textbook in the history of mankind
- The "Elements" contain only a single optimization problem

Problem

In a given triangle ABC, inscribe a parallelogram $ADEF$ (with $EF||AB$ and $DE||AC$) of maximal area.

- Solution: D, E, F are midpoints of the respective triangle sides

Steiner's Problem (1)

- Jakob Steiner (1796–1863):
  Swiss mathematician; worked primarily in geometry.
- Cavalieri (1598–1647), Torricelli (1608–1647), Viviani (1622–1703):
  Italian mathematicians and physicists

Problem

For a given triangle ABC,
find a point P that minimizes $|PA| + |PB| + |PC|$.

- Note: the minimizing point P is sometimes called *Steiner point*, sometimes *Torricelli point*, and sometimes *Fermat point*

Steiner's Problem (2)

Theorem

For a triangle ABC with all angles $\leq 120°$,
the Steiner point P satisfies $\angle APB = \angle BPC = \angle CPA = 120°$.

Theorem

For a triangle ABC with angle $\angle ABC \geq 120°$,
the Steiner point P coincides with point B.

Tartaglia's Problem (1)

Niccolo Tartaglia (1499/1500–1557)

- Italian mathematician, engineer, surveyor, bookkeeper
- founder of ballistics (paths of cannonballs)
- first to solve the cubic equation ($x^3 + ax^2 + bx + c = 0$)
- 1547/1548 conflict with Girolamo Cardano and Lodovico Ferrari
- six 'cartelli' and six 'risposte'

Ferrari's Problem 17 (in 3rd cartello)

Fatemi di 8 due tal parti, che'l prodotto dell'una nel altra moltiplicato nella loro differenza, faccia piu che possibil sia, dimostrando il tutto.

(Partition the number 8 into two parts, which yield the largest possible product when multiplied with each other and with their difference; and show me a proof.)

Tartaglia's Problem (2)

Modern language formulation

maximize $xy(x-y)$

such that $x+y=8$ and $x \geq y \geq 0$

Tartaglia's answer

Vi rispondo che la maggior parte fu 4 piu $R.(5+1/3)$ et la menore fu 4 meno $R.(5+1/3)$, el produtto é $10+2/3$, qual multiplicato nella differentia che é $R.(21+1/3)$ fa $R.(2427+7/27)$, et questa é di frutto della nostra pianta con li quali pensavati farmi guerra, ma el vi ha fallato el pensiero.

(I answer to you that the larger part is $4+4/\sqrt{3}$ and the smaller part is $4-4/\sqrt{3}$. The product $32/3$ multiplied by the difference $8/\sqrt{3}$ yields $256/\sqrt{27}$, and that's the result of my work. You wanted to defeat me, but your plans did not work out.)

Tartaglia's Problem (3)

An elementary solution

- Two parts $4 + x$ and $4 - x$ with $0 \leq x \leq 4$
  Goal: maximize $(4 + x)(4 - x)2x = 32x - 2x^3$
- Denote by M the maximum value that we are looking for
  Then $x^3 - 16x + \frac{1}{2}M = 0$

- We know that:
  the graph of $g(x) = x^3 - 16x + \frac{1}{2}M$ touches the x-axis,
  and has a double zero at the desired value of x, and a third zero
- Hence $g(x) = (x - a)^2(x - b)$
- Hence $x^3 - 16x + \frac{1}{2}M \ = \ x^3 - (2a + b)x^2 + (2ab + a^2)x - a^2 b$
- This yields $b = -2a$ and $3a^2 = 16$; and $a = 4/\sqrt{3}$ and $M = 256/\sqrt{27}$

Tartaglia's Problem (4)

But how did Tartaglia approach this problem?

- Tartaglia knew that the cubic equation $x^3 + px + q = 0$ has a solution

  $$\sqrt[3]{-q/2 + \sqrt{D}} \; + \; \sqrt[3]{-q/2 - \sqrt{D}}$$

  where $D = q^2/4 + p^3/27$
  (that's what we call Cardano's formula; it is due to Tartaglia)

- Tartaglia knew that sometimes something is going on at $D = 0$
- Natural move: solve $D = 0$ for $p = -16$ and $q = M/2$

Kepler's Problem (1)

Johannes Kepler (1571–1630)

- German mathematician, astronomer, and astrologer
- laws of planetary motion

In December of last year... I brought home a new wife at a time when Austria, having brought in a bumper crop of noble grapes, distributed its riches... The shore in Linz was heaped with wine barrels that sold at a reasonable price... That is why a number of barrels were brought to my house and placed in a row, and four days later the salesman came and measured all the tubes, without distinction, without paying attention to the shape, without any thought or computation. Namely the copper point of a ruler was pushed through the filling hole of a barrel, across the heel of each of the wooden disks which we refer to simply as bottoms, and as soon as the length to the point at the top of one board disk was the same as the length to the point at the bottom of the other, the salesman stated the number of amphoras contained in the barrel after merely noting the number on the ruler at the spot where the length in question ended. I was astonished...

Kepler's Problem (2)

Problem

Among all cylinders with the same space diagonal d,
find the one of maximal volume.

- Solution: ratio of base diameter to height equals $\sqrt{2}$

Some fundamental definitions

Definition

In a *continuous optimization problem*, we want to solve

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, r \\
 & h_i(x) = 0 & i = 1, \ldots, s \\
 & x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n &
\end{array}
$$

- The *dimension* of the problem is the number n of variables
- Recall: feasible point; feasible region; constraint; objective function
- Recall: local minimum; strict local minimum; global minimum
- Recall: linear function; linear optimization problem
- Recall: quadratic function; quadratic optimization problem
- Recall: non-linear optimization problem

Final example: A packing problem

Problem

Given: an integer $n \geq 1$

Compute: the largest radius $r = r(n)$,
such that n circles of radius r can be packed into unit square

$$
\begin{array}{lll}
\text{maximize} & r & \\
\text{subject to} & (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4r^2 & i,j = 1,\ldots,n, \quad i \neq j \\
& r \leq x_i & i = 1,\ldots,n \\
& r \leq y_i & i = 1,\ldots,n \\
& x_i \leq 1 - r & i = 1,\ldots,n \\
& y_i \leq 1 - r & i = 1,\ldots,n
\end{array}
$$

- Note: unsolved even for $n = 31$

Homework 1a

Recommended Exercises:
3, 6, 8, 9, 11, 12, 13

Collection of exercises can be downloaded from:
`http://www.win.tue.nl/∼gwoegi/optimization/`
`http://www.win.tue.nl/∼gwoegi/optimization/exer-1.pdf`