Optimization (2MMD10/2DME20), lecture 1b

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Our program for week 1

Tuesday:

- lots of examples from continuous optimization
  (mainly very old, mainly Greek, mainly geometric)

Friday:

- lots of examples from discrete optimization
  (mainly from the 20th century)
- some impossibility results

The Minimum Spanning Tree problem (1)

- 1920s: Electrification of South-West Moravia
- 1925/26: Jindřich Saxel, employee of Západomoravské elektrárny (West-Moravian Powerplants) contacts Otakar Borůvka
- Otakar Borůvka (1899–1995): Czech mathematician; Brno

Problem

Given n points in the plane,
join them by a net of minimum length
such that any two points are joined either directly or by means of some other points.

The Minimum Spanning Tree problem (2)

It is evident that a solution of this problem could have some importace in electricity power-line network design; hence I present the solution briefly using an example. The reader with a deeper interest in the subject is referred to the above quoted paper.

I shall give a solution of the problem in the case of 40 points given in Fig. 1. I shall join each of the given points with the nearest neighbor. Thus, for example, point 1 *with point* 2*, point* 2 *with point* 3*, point* 3 *with point* 4 *(point* 4 *with point* 3*), point* 5 *with point* 2*, point* 6 *with point* 5*, point* 7 *with point* 6*, point* 8 *with point* 9*, (point* 9 *with point* 8*), etc. I shall obtain a sequence of polygonal strokes* $1, 2, \ldots, 13$ *(Fig. 2).*

I shall join each of these strokes with the nearest stroke in the shortest possible way. Thus, for example, stroke 1 *with stroke* 2*, (stroke* 2 *with stroke* 1*), stroke* 3 *with stroke* 4*, (stroke* 4 *with stroke* 3*), etc. I shall obtain a sequence of polygonal strokes* $1, 2, \ldots, 4$ *(Fig. 3) I shall join each of these strokes in the shortest way with the nearest stroke. Thus stroke* 1 *with stroke* 3*, stroke* 2 *with stroke* 3 *(stroke* 3 *with stroke* 1*), stroke* 4 *with stroke* 1*. I shall finally obtain a single polygonal stroke (Fig. 4), which solves the given problem.*

The Minimum Spanning Tree problem (3)

Theorem

A minimum spanning tree can be found by the greedy algorithm.

Kruskal's algorithm:
Repeat "pick cheapest useful edge" until done

Prim's algorithm:
Grow the tree by repeatedly picking the cheapest edge between current tree and remaining points

Matroid = common generalization of cycle-free edge sets (graph theory) and independent sets of vectors (linear algebra)

The Soviet railway system problem (1)

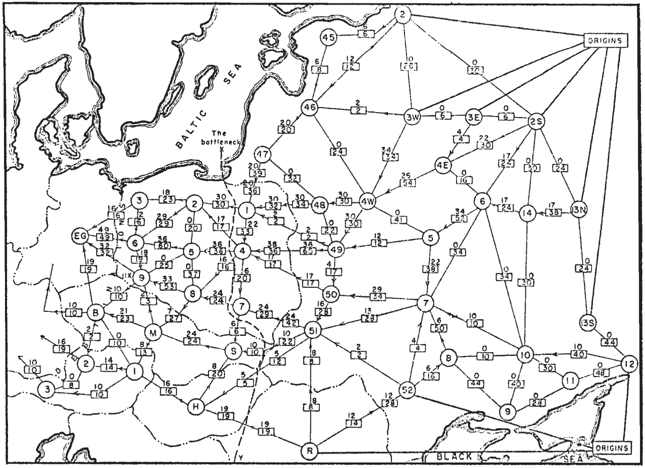


Fig. 2. From Harris and Ross [11]: Schematic diagram of the railway network of the Western Soviet Union and Eastern European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as "The bottleneck"

The Soviet railway system problem (2)

Problem

Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity.

Assuming a steady state condition, find a maximal flow from one given city to the other.

Problem (Harris & Ross, 1954)

Air power is an effective means of interdicting an enemy's rail system, and such usage is a logical and important mission.

As in many military operations, however, the success of interdiction depends largely on how complete, accurate, and timely is the commander's information, particularly concerning the effect of his interdiction-program efforts on the enemy's capability to move men and supplies. This information should be available at the time the results are being achieved.

The Soviet railway system problem (3)

Given:

- a directed network $G = (V, A)$
- source s and sink t in V
- arc capacities $c : A \to \mathbb{R}^+$

Definition

A flow is a mapping $f : A \to \mathbb{R}_0^+$ that satisfies
(a) $f(u, v) \leq c(u, v)$ for all arcs $(u, v) \in A$
(b) $\sum_{x:(x,u)\in A} f(x, u) \ = \ \sum_{y:(u,y)\in A} f(u, y)$ for all $u \in V - \{s, t\}$

The value $|f|$ of the flow is
$\sum_{x:(x,t)\in A} f(x, t)$ (and hence equals $\sum_{x:(s,x)\in A} f(s, x)$)

The Soviet railway system problem (4)

Given:

- a directed network $G = (V, A)$
- source s and sink t in V
- arc capacities $c : A \to \mathbb{R}^+$

Definition

An s-t cut (S, T) is a partition S and T of V
that satisfies $s \in S$ and $t \in T$.

The cut-set of (S, T) is the set $\{(u, v) \in A : \; u \in S, v \in T\}$.

The capacity of cut (S, T) is the total capacity of all arcs in the cut-set.

The Soviet railway system problem (5)

Easy observation

For an arbitrary flow f and for an arbitrary s-t cut (S, T) we have
value of flow $\leq$ capacity of cut.

Max-flow min-cut theorem

The value of the maximum flow
equals the capacity of the minimum cut.

- Lester R. Ford and Delbert R. Fulkerson (1955)

The Stigler diet problem (1)

- George Stigler (1911–1991); 1982 Nobel prize in economics; key leader of the Chicago School of Economics

Problem

For a moderately active man weighing 154 pounds,
how much of each of 77 foods should be eaten on a daily basis
so that the man's intake of nine nutrients will be at least equal to
the recommended dietary allowances (RDAs)
suggested by the National Research Council in 1943,
with the cost of the diet being minimal?

- In 1939, Stigler found a solution of cost \$39.93 (per year)
(Note: this would roughly correspond to \$600 nowadays.)

The Stigler diet problem (2)

Food	Annual Quantities	Annual Cost
Wheat Flour	370 lb.	$13.33
Evaporated Milk	57 cans	$3.84
Cabbage	111 lb.	$4.11
Spinach	23 lb.	$1.85
Dried Navy Beans	285 lb.	$16.80
Total Annual Cost		$39.93

Nutrient	Daily Recommended Intake
Calories	3,000 Calories
Protein	70 grams
Calcium	0.8 grams
Iron	12 milligrams
Vitamin A	5,000 IU
Thiamine (Vitamin B1)	1.8 milligrams
Riboflavin (Vitamin B2)	2.7 milligrams
Niacin	18 milligrams
Ascorbic Acid (Vitamin C)	75 milligrams

The Stigler diet problem (3)

- George Dantzig (1914–2005): American mathematical scientist; important contributions to operations research, computer science, economics, and statistics.
- Seven years after Stigler made his initial estimates, George Dantzig's Simplex algorithm found the exact optimal solution
- Optimal solution has cost \$39.69 (per year)
  "No one recommends these diets for anyone, let alone everyone."

The Stigler diet problem (4)

- The Stigler diet problem is a *linear program*

Definition

In a *linear program*, we want to solve

$$\begin{aligned} &\text{minimize} && f_0(x) && \\ &\text{subject to} && f_i(x) \le 0 && i = 1, \ldots, r \\ & && x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n && \end{aligned}$$

where f_0 and all f_i are linear functions

- Recall: LP duality
- Recall: Simplex algorithm
- Recall: Ellipsoid method solves LP in polynomial time

The Stigler diet problem (5a)

The great mathematical Sputnik of 1979

In England, the *Guardian* broke the story three days earlier, under the headline, "Soviet Answer to 'Traveling Salesmen.'" Choosing to emphasize the human side of the story, the *Guardian* said:

> "A young Soviet mathematician, apparently totally unknown to any of the world's senior practitioners, has found an answer to one of the most baffling problems in computer calculation.
>
> "But his obscurity is such that his discovery went unnoticed for 10 months in the mathematical world, although work on the problem has been going on for years.
>
> "The apparent breakthrough was achieved by L. G. Khachian, and was published in a Soviet journal, *Doklady*, last January. Few people in the West read the journal and it was only after rumours of the discovery circulated at a conference in Germany that anyone in the mathematical world at large had even a hint that someone had come up with an answer to what is known in the trade as the 'Traveling Salesman' problem."

The Stigler diet problem (5b)

The *Times* story appears to have been based on certain unshakable preconceptions of its writer, Malcolm W. Browne. Browne called George Dantzig of Stanford University, a great pioneering authority on linear programming, and tried to force him into various admissions. Dantzig's version of the interview bears repeating.

> "What about the traveling salesman problem?" asked Browne. "If there is a connection, I don't know what it is," said Dantzig. ("The Russian discovery proposed an approach for [solving] a class of problems related to the 'Traveling Salesman Problem,'" reported Browne.) "What about cryptography?" asked Browne. "If there is a connection, I don't know what it is," said
>
> Dantzig. ("The theory of codes could eventually be affected," reported Browne.) "Is the Russian method practical?" asked Browne. "No," said Dantzig. ("Mathematicians describe the discovery ... as a method by which computers can find solutions to a class of very hard problems that has hitherto been attacked on a hit-or-miss basis," reported Browne.)

Cliques and Matchings in social networks (1)

- Facebook friendship graph:
  vertices=people; edges=friendships

Definition

A group S of vertices forms a clique,
if any two vertices $u, v \in S$ are connected by an edge

Definition

A group S of vertices contains a perfect matching,
if S can be partitioned into pairs (groups of size 2)
that each are connected by an edge

- Recall: fast algorithm for finding perfect matchings

The Travelling Salesman Problem (1)

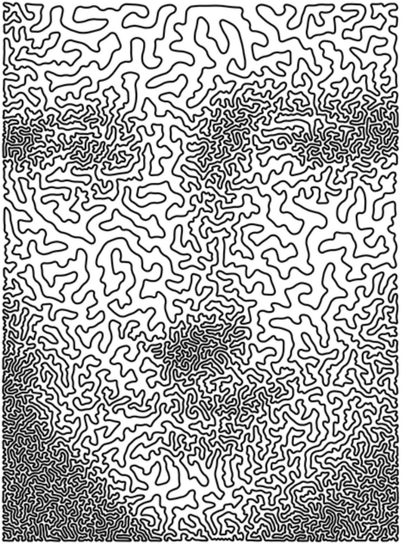
Travelling Salesman Problem (TSP)

Given: n cities; distances $d(i, j)$ between cities i and j

Compute: the shortest round-trip through all cities

- The travelling salesman starts in a city, visits all other cities, and finally returns to his starting point; Goal = minimize gas
- TSP models many hard real-world problems
  Merrill Flood (1930): school bus routing
  VLSI microchip layout problem
  Change-over times in machine scheduling

The Travelling Salesman Problem (2)



Some fundamental definitions

Definition

In an *integer* optimization problem, we want to solve

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, r \\
& h_i(x) = 0 & i = 1, \ldots, s \\
& x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n &
\end{array}
$$

- Recall: linear function; linear optimization problem
- Recall: Integer linear program (ILP)
- Recall: quadratic function; quadratic optimization problem
- Recall: non-linear optimization problem

Hardness and undecidability (1)

$$\begin{array}{lll} (1) & \text{minimize} & x^3 + y^3 + z^3 \\ & \text{subject to} & x^3 + y^3 + z^3 \geq 29 \\ & & x, y, z \in \mathbb{Z} \end{array}$$

$$\begin{array}{lll} (2) & \text{minimize} & x^3 + y^3 + z^3 \\ & \text{subject to} & x^3 + y^3 + z^3 \geq 30 \\ & & x, y, z \in \mathbb{Z} \end{array}$$

$$\begin{array}{lll} (3) & \text{minimize} & x^3 + y^3 + z^3 \\ & \text{subject to} & x^3 + y^3 + z^3 \geq 33 \\ & & x, y, z \in \mathbb{Z} \end{array}$$

- Solution to (1): $(x, y, z) = (3, 1, 1)$ with value 29
- Solution to (2): $(x, y, z) = (-283059965, -2218888517, 2220422932)$ with value 30
- Solution to (3): nobody knows

Hardness and undecidability (2)

The cattle problem of Archimedes

Story about milk-white, black, dappled and yellow cows that leads to the equation $x^2 = 410286423278424 \cdot y^2 + 1$.

In the smallest positive solution, x has 206545 digits

Theorem (Andrew Wiles, 1995)

The optimal objective value of the following problem is not zero.

$$\begin{array}{rl} \text{minimize} & (x^n + y^n - z^n)^2 \\ \text{subject to} & x, y, z, n \in \mathbb{Z} \\ & x, y, z \geq 1 \;\text{ and }\; n \geq 3 \end{array}$$

This has first been conjectured by Pierre de Fermat in 1637

Hardness and undecidability (3)

Hilbert's Tenth Problem

Given a polynomial $P(x_1, x_2, \ldots, x_n)$ with coefficients in $\mathbb{Z}$,
decide whether there exist integers $x_1, x_2, \ldots, x_n$
such that $P(x_1, x_2, \ldots, x_n) = 0$

Theorem (highschool knowledge)

The case $n = 1$ has an easy algorithmic solution.
(check all divisors of the constant term)

Theorem (Matiyasevich, 1970)

There is no algorithm that solves Hilbert's Tenth Problem.

Hardness and undecidability (4)

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \le 0 & i = 1, \ldots, r \\
& h_i(x) = 0 & i = 1, \ldots, s \\
& x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n &
\end{array}
$$

Consequence

There is no algorithm that solves
optimization problems over the integers.

Consequence

There is no algorithm that solves
optimization problems with polynomial objective function
and polynomial constraints over the integers.

Hardness and undecidability (5)

Two positive results:

Theorem

There is an algorithm that solves
optimization problems with linear objective function
and linear constraints over the integers.

Theorem (Tarski, 1951)

There is an algorithm that solves
optimization problems over the reals.

An unsolvable algorithmic problem

Problem: CheckTermination

Input: two text pieces `text.1` and `text.2`

Question: does the C++ program listed in `text.1` terminate on the input in `text.2`?

- Suppose there exists an algorithm for CheckTermination
- Then there exists a C++ program that implements this algorithm

- We construct a new C++ program XXX that takes input `text.3`
- First, XXX checks whether the C++ program listed in `text.3` terminates on the input in `text.3`
- If `text.3` does terminate, then XXX goes into infinite loop
- If `text.3` does not terminate, then XXX stops

QUESTION:
What does XXX do, if we input the C++ code of XXX to it?

Another unsolvable algorithmic problem

Problem: Post Correspondence Problem

Input: two sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of strings

Question: does there exist a finite sequence $s(1), s(2), \ldots, s(m)$
such that the concatenation $x_{s(1)} x_{s(2)} \cdots x_{s(m)}$
equals the concatenation $y_{s(1)} y_{s(2)} \cdots y_{s(m)}$?

Example

$n = 3$, and
$x_1 = a$, $x_2 = ab$, $x_3 = bba$, and
$y_1 = baa$, $y_2 = aa$, $y_3 = bb$.

Homework 1b

Recommended Exercises:
15, 16, 19, 20, 22, 23, 24

Collection of exercises can be downloaded from:
`http://www.win.tue.nl/∼gwoegi/optimization/`
`http://www.win.tue.nl/∼gwoegi/optimization/exer-1.pdf`

Attention!
Weeks 2-4 (Sep 8; Sep 15; Sep 22; Sep 29):

- Tuesday 1+2: instructions
- Tuesday 3+4: lecture