Exam "Nonlinear Optimization (2DE09)" TU/e
Wednesday, 5 november 2014, 9:00–12:00

There are 8 questions, each worth a total of 4 points. You are **not allowed** *to use any tools other than pen and paper: no books, no notes, no pocket calculators!*

(1) Are the following sets convex? Prove.
(a) $\{1, 2, 3, 4, 5, 6\}$
(b) $\{x \in \mathbb{R}^3 \mid \|x\| \geq 1\}$
(c) $\{\begin{bmatrix} 3x + y \\ -y - 2 \end{bmatrix} \mid x \geq 0,\ y \leq 0,\ x^2 + y^2 \leq 1\}$
(d) The set of symmetric real 4×4 matrices whose smallest eigenvalue satisfies $\lambda_{\min} \leq 1$

(2) Let $K = \{A \in S^n \mid x^T Ax \geq 0 \text{ for all } x \geq 0\}$ and $L = \{B \in S^n \mid B = yy^T,\ y \geq 0\}$.
(a) Give a matrix in K that is not positive semi-definite.
(b) Prove that K is the dual cone L^* of L.
(c) Prove that cone K has non-empty interior and is pointed.

(3) Are the following functions convex on the given domain? Prove.
(a) $f(x) = \arctan(x)$ on $\mathbb{R}_{++}$
(b) $f(x, y, z) = (y + 2z)^2/(x - 3y)$ on $\{(x, y, z) \in \mathbb{R}^3 \mid x - 3y > 0\}$
(c) $f(x, y, z) = x^2 + y^2 + 5z^2 - xy - xz - 3yz$ on $\mathbb{R}^3$
(d) $f(x) = \max\{\|x\|,\ \|x - a\|\}$ on $\mathbb{R}^n$, where $a \in \mathbb{R}^n$

(4) For $P, Q \in S^n$ consider the primal problem $\min\{x^T Px \mid 2 \leq x^T Qx \leq 3,\ x^T x = 1\}$.
(a) Give the Lagrangian, Lagrange dual function, and Lagrange dual problem.
(b) Show that the dual is equivalent to a semi-definite optimization problem.

(5) For vector $a \in \mathbb{R}^3$, matrices $B_0, B_1, B_2, B_3 \in S^n$, and $c \in \mathbb{R}$ consider the primal problem $\min\{a^T x \mid B_0 + x_1 B_1 + x_2 B_2 + x_3 B_3 \succeq 0,\ x_1 + x_2 + x_3 = c,\ x \in \mathbb{R}^3\}$.
Give the Lagrangian, Lagrange dual function, and Lagrange dual problem.

(6) Let $A_0, A_1, \ldots, A_k$, B, C be symmetric $n \times n$ matrices and let $q \in \mathbb{R}$. We want to find a matrix A in the set $\{A_0 + x_1 A_1 + \cdots + x_k A_k \mid x \in \mathbb{R}^k\}$ such that no eigenvalue of BA exceeds the bound q and such that the smallest eigenvalue of CA is as large as possible. Formulate this problem as a semi-definite optimization problem.

(7) Are the three functions in (a)–(c) self-concordant? Prove.
(a) $f(x) = x^6/30$ on $\mathbb{R}$
(b) $f(x) = x \log x - 8 \log x$ on $\mathbb{R}_{++}$
(c) $f(x) = -\log(x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2)$ on $\mathbb{R}^4$
(d) Prove or disprove: If f is self-concordant on $\mathbb{R}$, then $\frac{1}{2}f$ is self-concordant on $\mathbb{R}$.

(8) (a) Define when a search direction Δx is a *descent direction*.
(b) Show that for strictly convex f, the Newton direction Δx_{nt} is a descent direction. (Recall that $\Delta x_{nt} = -(\nabla^2 f(x))^{-1} \nabla f(x)$.)
(c) Define the *degree* of a generalized logarithm for a cone K.
(d) State (without proof) a generalized logarithm for the semi-definite cone.

Point distribution: (1) 1,1,1,1 (2) 1,2,1 (3) 1,1,1,1 (4) 3,1 (5) 4 (6) 4 (7) 1,1,1,1 (8) 1,1,1,1