Eindhoven University of Technology
Department of Mathematics and Computer Science
Nonlinear Optimization (2DE09)
Monday 7 april 2014, 14:00 - 17:00

There are 10 questions, worth 1 point each. If you have earned any bonus points, then indicate clearly which questions you want to drop. You are not allowed to use any tools other than pen, paper and a pocket calculator.

(1) Are the following sets convex? Prove.
  (a) $\{x \in \mathbb{R} \mid \cos(x) \leq 0\}$
  (b) $\{(x, y) \in \mathbb{R}^2 \mid x + 2y \leq 10,\ 5x + y \leq 4\}$
  (c) $\{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 4$ or $(x+1)^2 + y^2 \leq 4\}$
  (d) $\{(x_1, x_2) \mid x_1 + x_2 + x_3 \geq 0,\ x_1^2 + 2x_2^2 + 3x_3^2 \leq 1,\ x_1, x_2, x_3 \in \mathbb{R}\}$

(2) Are the following sets convex? Prove.
  (a) $\{(a, b, c) \in \mathbb{R}^3 \mid ax^3 + bx - c \geq 4$ for $x \in [4, 5]\}$
  (b) $\{A \in S^k \mid \lambda_1(A) \geq 1\}$
  (S^k is the set of real $k \times k$ symmetric matrices, and $\lambda_1(A)$ is the smallest eigenvalue of A)

(3) Are the following functions convex on the given domain? Prove.
  (a) $f(x) = x \log(x)$ on $(0, \infty)$
  (b) $f(x, y) = 2^x - \log(y) + \exp(x + y)$ on $(0, \infty)^2$
  (c) $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 3xy + 2yz$ on $\mathbb{R}^3$
  (d) $f(x, y, z) = -\log(x + y + z)$ on $(0, \infty)^3$

(4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Show that f is a convex function if and only if the function $g_{x,d}(\lambda) := f(x + \lambda d)$ is convex for each $x, d \in \mathbb{R}^n$

(5) Give the Lagrangian, Lagrange dual function, and Lagrange dual of

$$\min\{x^T P x \mid a^T x \geq b,\ x \geq 0,\ x^T x = 1,\ x \in \mathbb{R}^n\},$$

where $P \in S^k$ is a positive definite matrix, $a \in \mathbb{R}^n, b \in \mathbb{R}$.

(6) Give the Lagrangian, Lagrange dual function, and Lagrange dual of the following semidefinite optimization problem:

$$\min\{c^T x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0,\ B_0 + x_1 B_1 + \cdots + x_n B_n \succeq 0,\ x \in \mathbb{R}^n\}.$$

Here $A_0, \ldots, A_n, B_0, \ldots, B_n \in S^k$.

(7) Derive the dual of the program

$$\min\{t \mid \begin{bmatrix} tI & A(x) \\ A(x) & tI \end{bmatrix} \succeq 0, A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}\}$$

where $A_0, \ldots, A_n \in S^k$.

(8) Consider the semidefinite optimization problem

$$\min\{c^T x \mid x_1 A_1 + \cdots + x_n A_n \preceq A_0,\ x \in \mathbb{R}^n\} \tag{1}$$

where $A_0, \ldots, A_n \in S^k$ and $c \in \mathbb{R}^n$. Suppose that there is a nonsingular $k \times k$ matrix R such that $R^T A_i R$ is diagonal for $i = 0, \ldots, n$. Show that (1) is equivalent to a linear program.

(9) State when a function $f : \mathbb{R} \to \mathbb{R}$ is self-concordant on a given domain. Find the smallest $\alpha > 0$ such that $f(x) := \alpha/x$ is self-concordant on the domain $(0, 8/9)$.

(10) State when a real-valued function f of several variables is self-concordant on a given domain. Prove that the function $f(x, y) = -\log(y^2 - x^T x)$ is self-concordant on $\{(x, y) \mid x^T x < y^2, x \in \mathbb{R}^n, y \in \mathbb{R}\}$.