2DME20 Exam "Non-linear Optimization"

Friday, 30 october 2015 **TU/e**

There are 8 questions worth a total of 80 points. You are **not allowed** *to use any tools other than pen and paper: no books, no notes, no pocket calculators!*

(1) (a) Formulate the arithmetic-geometric mean inequality for $n = 3$ positive real variables.
(b) Determine the minimum of the following function $f(a,b,c,d)$ for $a,b,c,d \in \mathbb{R}^+$.

$$f(a,b,c,d) \;=\; \frac{a^8+2b}{b+c+d} + \frac{b+c+d}{abc+2d} + \frac{abc+2d}{a^8+2b}$$

(2) Let $K = \{x \mid y^T x \geq 0 \text{ for all } y \in \mathbb{R}^n_+\}$.
(a) Prove that K is a convex set.
(b) Prove or disprove: $K = \mathbb{R}^n_+$

(3) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = 6x^2 + y^2 - 12xy$.
(a) Decide whether function f is convex.
(b) Give the KKT conditions for optimizing $f(x,y)$ subject to the constraint $x^2 + y^2 = 13$.
(c) Determine the maximum and the minimum of $f(x,y)$ subject to $x^2 + y^2 = 13$.

(4) For $A, B, C \in S^n_+$ consider the problem $\min\{x^T Ax \mid x^T Bx \leq 3,\; x^T Cx \leq 5,\; x^T x = 1\}$.
(a) Give the Lagrangian, the Lagrange dual function and the Lagrange dual.
(b) Show that the dual is equivalent to a semi-definite optimization problem.

(5) An instance of MATRIX-VALUE consists of an $n \times n$ matrix A with integer entries and an integer value t. The question is whether there exists a set S of n entries in the matrix, so that S contains exactly one entry from each row and each column, and so that the entries in S add up to the value t.
(a) Prove that MATRIX-VALUE lies in NP.
(b) Prove that MATRIX-VALUE is NP-hard.

(6) (a) State the ILP formulation for VERTEX COVER.
(b) Prove that the integrality gap of the LP relaxation is at least 2.

(7) (a) Define the *degree* of a generalized logarithm for a cone K
(b) State (without proof) generalized logarithms for the non-negative orthant cone, for the second-order cone, and for the semi-definite cone.
(c) State (without proof) the degrees of the three generalized logarithms under (b).

(8) Reformulate the following problem into a second-order cone optimization problem with linear objective function and constraints that are either linear or SOC-constraints.

$$\begin{aligned} \text{minimize} \quad & \max\{x - 4y + 9z + 20,\; \frac{(z-x)^2 + 2y^2}{1+x+y}\} \\ \text{subject to} \quad & x^2 + y^2 - 3x + 5y \;\leq\; 23z + 6 \\ & x, y, z \geq 0 \end{aligned}$$

Point distribution: (1) 3,7 (2) 5,5 (3) 3,3,4 (4) 7,3 (5) 2,8 (6) 5,5 (7) 2,5,3 (8) 10