The Freudenthal problem and its ramifications (Part I)

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Abstract

This is the first article (in a series of three) dedicated to the many variants and variations of the so-called Freudenthal problem. The Freudenthal problem is a mathematical puzzle in which the reader deduces two secret integers from several rounds of communication between two persons. One person knows the sum of the two secret integers, while the other person knows the product. The current article surveys some of the most basic variants of the Freudenthal problem.

1 The Freudenthal problem

Hans Freudenthal (1905-1990) studied mathematics at the University of Berlin in the 1920s. He completed his Ph.D. thesis *"Über die Enden topologischer Räume und Gruppen"* under the supervision of Heinz Hopf in 1930. Around that time, he moved to the Netherlands where he worked with Luitzen Brouwer and soon became a lecturer at the University of Amsterdam. As a Jew, Freudenthal survived the period of German occupation unharmed, since he was married to an Arian Dutch woman and since he had lots of luck. In 1946, Freudenthal was offered the chair of pure and applied mathematics at the University of Utrecht. He held this chair until he retired in 1975. Freudenthal's scientific contributions mainly fall into topology, geometry, and the theory of Lie groups. Freudenthal is also remembered and recognized for his numerous contributions to mathematical education and didactics. The institute for innovation and improvement of mathematics education at the University of Utrecht is named after him the *"Freudenthal Institute"*.

In 1969, Hans Freudenthal [2] posed the following puzzle in the problem section of the Dutch mathematics journal *Nieuw Archief voor Wiskunde* (= New

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archive for mathematics). The original formulation of the puzzle is in Dutch. Here is our free translation:

> The teacher says to Peter and Sam: I have secretly chosen two integers x and y with $2 \leq x < y$ and $x + y \leq 100$. I have told the sum $s = x + y$ to Sam (but not to Peter) and the product $p = xy$ to Peter (but not to Sam).
>
> 1. Peter says: I don't know the numbers x and y.
> 2. Sam replies: I already knew you didn't know.
> 3. Peter says: Oh, then I do know the numbers x and y.
> 4. Sam says: Oh, then I also know them.
>
> Determine x and y!

At first sight, the given information just cannot be enough for determining the two numbers... The Freudenthal problem was introduced to the English speaking world in 1976, when David Sprows stated it in the problem section of the Mathematics Magazine [7]. In December 1979, Martin Gardner [4] posed the Freudenthal problem in his mathematical entertainments column in the Scientific American. He writes: *"I call this beautiful problem impossible, because it seems to lack sufficient information for a solution."* And indeed, nowadays the Freudenthal problem sometimes shows up under the name *"The impossible problem"*; see for instance Sallows [6]. Edsger Dijkstra [1] reports that he once solved a variant of the Freudenthal problem during a sleepless night in 1978, when he was jet-lagged. He states that it took him almost six hours, and that he solved it in his head, without using paper or pencil.

In this article, we want to discuss some of the most basic Freudenthal variants. We will mainly concentrate on two classes of variants, that are built around the following definitions. Consider two positive integers m and M with $m \leq M$, and define the following sets:

> $Z^{\neq}(m, M)$ contains all pairs (x, y) with $m \leq x < y$ and $x + y \leq M$.
>
> $Z^{=}(m, M)$ contains all pairs (x, y) with $m \leq x \leq y$ and $x + y \leq M$.

In the Freudenthal variant $\text{FREUDENTHAL}^{\neq}(m, M)$ the introductory words of the teacher state that the secret number pair (x, y) is taken from $Z^{\neq}(m, M)$. In the Freudenthal variant $\text{FREUDENTHAL}^{=}(m, M)$ the introductory words of the teacher state that the secret number pair (x, y) is taken from $Z^{=}(m, M)$. In both variants, the announcement of the teacher is followed by the above four-line conversation between Sam and Peter. Note that the problem originally posed by Hans Freudenthal is $\text{FREUDENTHAL}^{\neq}(2, 100)$.

2 The algorithm of Denniston

The *Nieuw Archief voor Wiskunde* [3] lists the names of seventeen readers who submitted correct solutions for the Freudenthal problem; interestingly, two of the names on this list are *J. van Leeuwen* and *J.H. van Lint*. Among other solutions, [3] discusses a simple computational approach by Ralph Hugh Francis Denniston. Although we will only formulate Denniston's approach for problem FREUDENTHAL$^{\neq}(m, M)$, it obviously generalizes to other Freudenthal variants.

Initialization. Introduce a matrix A where the rows p correspond to the products and the columns s correspond to the sums.
Set entry $A[p, s]$ to $+$, if there exist integers x, y with $(x, y) \in Z^{\neq}(m, M)$ that satisfy $x + y = s$ and $xy = p$. Otherwise, set $A[p, s]$ to $-$.

Step 1. Wherever a row p contains just a single + entry, replace this entry by 1. (This product p contradicts statement #1 by Peter.)

Step 2. Wherever a column s contains some 1 entry, replace all + entries in this column by 2. (This sum s contradicts statement #2 by Sam.)

Step 3. Wherever a row p contains two or more + entries, replace them by 3. (This product p contradicts statement #3 by Peter.)

Step 4. Wherever a column s contains two or more + entries, replace them by 4. (This sum s contradicts statement #4 by Sam.)

Output. The remaining + entries specify all sum/product combinations that agree with the full conversation. A + entry in row p_0 and column s_0 means that the values s_0 and p_0 are sum and product of the secret numbers x and y.

If in the end there is a single remaining + entry, then the Freudenthal problem has a unique solution. If there is more than one remaining + entry, then the problem has several possible solutions; Sam and Peter are able to determine x and y from the conversation (and from their private knowledge of s or p), whereas the reader is not. If there are no remaining + entries, then the problem formulation is contradictory.

Some more notation: We say that a sum s and a product p are *compatible* (with respect to some fixed Freudenthal problem that usually is clear from the context), if the initialization step of Denniston's algorithm sets entry $A[p, s]$ to $+$. During an execution of Denniston's algorithm, a row or a column is called *alive* if it contains at least one + entry. We denote by $\mathcal{P}_1$ the set of rows/products p that are alive after Step 1; note that these products are in agreement with statement #1. Similarly, we denote by $\mathcal{S}_2$ the set of columns/sums s that are alive after Step 2 (and that agree with statements #1 and #2), we denote by $\mathcal{P}_3$ the set of rows/products p that are alive after Step 3 (and that agree with statements #1, #2, and #3), and we denote by $\mathcal{S}_4$ the set of columns/sums s that are alive after Step 4 (and that agree with the full conversation).

3 The Freudenthal problem with m=1 and M=11

We now take a closer look at $\text{Freudenthal}^{\neq}(1, 11)$ and $\text{Freudenthal}^{=}(1, 11)$, which behave surprisingly different from each other.

Table 1 summarizes Denniston's algorithm for $\text{Freudenthal}^{\neq}(1, 11)$. This puzzle is contradictory and ill-posed: Statement #1 yields $\mathcal{P}_1 = \{6, 8, 10, 12, 18, 24\}$, and statement #2 gives $\mathcal{S}_2 = \{7\}$. In statement #3, Peter determines x and y from his product p and from $s = 7$. This makes $\mathcal{P}_3 = \{6, 10, 12\}$, and leaves us with the three possibilities $(1, 6)$, $(2, 5)$, and $(3, 4)$ for (x, y). Sam cannot make statement #4, as there is no way for him to identify the correct product from $s = 7$ and $\mathcal{P}_3$.

Table 2 demonstrates that problem $\text{Freudenthal}^{=}(1, 11)$ is well-posed and has a unique solution. Since $x = y$ is legal in this variant, statement #1 now yields $\mathcal{P}_1 = \{4, 6, 8, 9, 10, 12, 16, 18, 24\}$. Statement #2 restricts the sum to $\mathcal{S}_2 = \{5, 7\}$. In statement #3, Peter determines x and y from his product: The product cannot be 6, since then Peter could not distinguish $x = 2$, $y = 3$, $s = 5$ from $x = 1$, $y = 6$, $s = 7$. Therefore $\mathcal{P}_3 = \{4, 10, 12\}$. Finally, statement #4 implies $s \neq 7$, since otherwise Sam could not distinguish between $p = 10$ and $p = 12$. Therefore $s = 5$ and $p = 4$, which yields $x = 1$ and $y = 4$.

4 An analysis of the classical Freudenthal problem

We now want to get some understanding how Denniston's algorithm behaves for the classical Freudenthal problem $\text{Freudenthal}^{\neq}(2, 100)$.

The set $\mathcal{P}_1$ is listed in Table 3; it consists of 574 elements, but has a rather primitive structure: Every element $p \in \mathcal{P}_1$ possesses at least two factorizations $p = xy$ with $(x, y) \in Z^{\neq}(2, 100)$. Here are some simple rules for excluding certain products from $\mathcal{P}_1$: First, any product of two prime numbers is not in $\mathcal{P}_1$. Secondly, any p with a prime factor greater than 50 is not in $\mathcal{P}_1$. Next, any number of the form $p = q^3$ with prime q is not in $\mathcal{P}_1$; otherwise, Peter would deduce $x = q$ and $y = q^2$ right at the beginning. Finally, any number of the form $p = 2q^2$ with a prime $q > 10$ is not in $\mathcal{P}_1$; otherwise, Peter could deduce $x = q$ and $y = 2q$.

Next, let us investigate the structure of set $\mathcal{S}_2$. For $s \in \mathcal{S}_2$, all compatible products $x(s-x)$ must lie in $\mathcal{P}_1$. Hence, the following values of s are not contained in $\mathcal{S}_2$:

$p\backslash s$	2	3	4	5	6	7	8	9	10	11
1	–	–	–	–	–	–	–	–	–	–
2	–	1	–	–	–	–	–	–	–	–
3	–	–	1	–	–	–	–	–	–	–
4	–	–	–	1	–	–	–	–	–	–
5	–	–	–	–	1	–	–	–	–	–
6	–	–	–	2	–	4	–	–	–	–
7	–	–	–	–	–	–	1	–	–	–
8	–	–	–	–	2	–	–	2	–	–
9	–	–	–	–	–	–	–	–	1	–
10	–	–	–	–	–	4	–	–	–	2
11	–	–	–	–	–	–	–	–	–	–
12	–	–	–	–	–	4	2	–	–	–
13	–	–	–	–	–	–	–	–	–	–
14	–	–	–	–	–	–	–	1	–	–
15	–	–	–	–	–	–	1	–	–	–
16	–	–	–	–	–	–	–	–	1	–
17	–	–	–	–	–	–	–	–	–	–
18	–	–	–	–	–	–	–	2	–	2
19	–	–	–	–	–	–	–	–	–	–
20	–	–	–	–	–	–	–	1	–	–
21	–	–	–	–	–	–	–	–	1	–
22	–	–	–	–	–	–	–	–	–	–
23	–	–	–	–	–	–	–	–	–	–
24	–	–	–	–	–	–	–	–	2	2
25	–	–	–	–	–	–	–	–	–	–
26	–	–	–	–	–	–	–	–	–	–
27	–	–	–	–	–	–	–	–	–	–
28	–	–	–	–	–	–	–	–	–	1
29	–	–	–	–	–	–	–	–	–	–
30	–	–	–	–	–	–	–	–	–	1

Table 1: The outcome of Denniston's algorithm for FREUDENTHAL$^{\neq}(1, 11)$.

$p \backslash s$	2	3	4	5	6	7	8	9	10	11
1	1	–	–	–	–	–	–	–	–	–
2	–	1	–	–	–	–	–	–	–	–
3	–	–	1	–	–	–	–	–	–	–
4	–	–	2	+	–	–	–	–	–	–
5	–	–	–	–	1	–	–	–	–	–
6	–	–	–	3	–	3	–	–	–	–
7	–	–	–	–	–	–	1	–	–	–
8	–	–	–	–	2	–	–	2	–	–
9	–	–	–	–	2	–	–	–	2	–
10	–	–	–	–	–	4	–	–	–	2
11	–	–	–	–	–	–	–	–	–	–
12	–	–	–	–	–	4	2	–	–	–
13	–	–	–	–	–	–	–	–	–	–
14	–	–	–	–	–	–	–	1	–	–
15	–	–	–	–	–	–	1	–	–	–
16	–	–	–	–	–	–	2	–	2	–
17	–	–	–	–	–	–	–	–	–	–
18	–	–	–	–	–	–	–	2	–	2
19	–	–	–	–	–	–	–	–	–	–
20	–	–	–	–	–	–	–	1	–	–
21	–	–	–	–	–	–	–	–	1	–
22	–	–	–	–	–	–	–	–	–	–
23	–	–	–	–	–	–	–	–	–	–
24	–	–	–	–	–	–	–	–	2	2
25	–	–	–	–	–	–	–	–	1	–
26	–	–	–	–	–	–	–	–	–	–
27	–	–	–	–	–	–	–	–	–	–
28	–	–	–	–	–	–	–	–	–	1
29	–	–	–	–	–	–	–	–	–	–
30	–	–	–	–	–	–	–	–	–	1

Table 2: The outcome of Denniston's algorithm for FREUDENTHAL$^=(1, 11)$.

12,	18,	20,	24,	28,	30,	32,	36,	40,	42,
44,	45,	48,	50,	52,	54,	56,	60,	63,	64,
66,	68,	70,	72,	75,	76,	78,	80,	84,	88,
90,	92,	96,	98,	99,	100,	102,	104,	105,	108,
110,	112,	114,	116,	117,	120,	124,	126,	128,	130,
132,	135,	136,	138,	140,	144,	147,	148,	150,	152,
153,	154,	156,	160,	162,	164,	165,	168,	170,	171,
172,	174,	175,	176,	180,	182,	184,	186,	188,	189,
190,	192,	195,	196,	198,	200,	204,	207,	208,	210,
216,	220,	222,	224,	225,	228,	230,	231,	232,	234,
238,	240,	243,	245,	246,	248,	250,	252,	255,	256,
258,	260,	261,	264,	266,	270,	272,	273,	275,	276,
279,	280,	282,	285,	286,	288,	290,	294,	296,	297,
300,	304,	306,	308,	310,	312,	315,	320,	322,	324,
325,	328,	330,	336,	340,	342,	344,	345,	348,	350,
351,	352,	357,	360,	364,	368,	370,	372,	374,	375,
376,	378,	380,	384,	385,	390,	392,	396,	399,	400,
405,	406,	408,	410,	414,	416,	418,	420,	425,	429,
430,	432,	434,	435,	440,	441,	442,	444,	448,	450,
455,	456,	459,	460,	462,	464,	465,	468,	470,	475,
476,	480,	483,	486,	490,	492,	494,	495,	496,	500,
504,	506,	510,	512,	513,	516,	518,	520,	522,	525,
528,	532,	539,	540,	544,	546,	550,	552,	558,	560,
561,	564,	567,	570,	572,	574,	576,	580,	585,	588,
592,	594,	595,	598,	600,	602,	608,	609,	612,	616,
620,	621,	624,	627,	630,	637,	638,	640,	644,	646,
648,	650,	651,	656,	660,	663,	666,	672,	675,	680,
682,	684,	688,	690,	693,	696,	700,	702,	704,	714,
715,	720,	726,	728,	735,	736,	738,	740,	741,	744,
748,	750,	754,	756,	759,	760,	765,	768,	770,	774,
780,	782,	783,	784,	792,	798,	800,	806,	810,	812,
814,	816,	819,	820,	825,	828,	832,	836,	840,	850,
855,	858,	860,	864,	868,	870,	874,	880,	882,	884,
888,	891,	896,	897,	900,	902,	910,	912,	918,	920,
924,	928,	930,	935,	936,	945,	946,	950,	952,	957,
960,	962,	966,	968,	969,	972,	975,	980,	984,	986,
988,	990,	992,	1000,	1008,	1012,	1014,	1020,	1026,	1032,
1035,	1036,	1040,	1044,	1050,	1053,	1054,	1056,	1064,	1066,
1071,	1078,	1080,	1088,	1092,	1100,	1102,	1104,	1105,	1110,
1116,	1118,	1120,	1122,	1125,	1131,	1134,	1140,	1144,	1148,
1150,	1152,	1155,	1160,	1170,	1173,	1176,	1178,	1184,	1188,
1190,	1196,	1197,	1200,	1204,	1215,	1216,	1218,	1224,	1230,
1232,	1240,	1242,	1248,	1254,	1258,	1260,	1275,	1276,	1280,
1288,	1292,	1296,	1300,	1302,	1311,	1312,	1320,	1323,	1326,
1330,	1332,	1334,	1344,	1350,	1360,	1364,	1365,	1368,	1377,
1380,	1386,	1392,	1394,	1400,	1404,	1406,	1408,	1425,	1426,
1428,	1430,	1440,	1449,	1450,	1452,	1456,	1458,	1470,	1472,
1476,	1480,	1482,	1485,	1488,	1496,	1500,	1508,	1512,	1518,
1520,	1530,	1536,	1539,	1540,	1550,	1554,	1560,	1564,	1566,
1568,	1575,	1584,	1596,	1600,	1610,	1612,	1617,	1620,	1624,
1628,	1632,	1638,	1650,	1656,	1664,	1672,	1674,	1680,	1700,
1702,	1710,	1716,	1725,	1728,	1736,	1740,	1748,	1750,	1755,
1760,	1764,	1768,	1776,	1782,	1792,	1794,	1798,	1800,	1820,
1824,	1836,	1848,	1850,	1856,	1860,	1872,	1890,	1904,	1914,
1920,	1924,	1932,	1938,	1944,	1950,	1960,	1972,	1980,	1984,
2016,	2030,	2040,	2046,	2052,	2070,	2080,	2100,	2108,	2112,
2142,	2145,	2160,	2176,	2184,	2200,	2205,	2240,	2244,	2268,
2280,	2340,	2352,	2400.						

Table 3: The set $\mathcal{P}_1$ for FREUDENTHAL$^{\neq}(2, 100)$.

- $55 \leq s \leq 100$: For $x = 53$ and $y = s - 53$, the product xy is not in $\mathcal{P}_1$.
- $s = 6$: The product of $x = 2$ and $y = 4$ is not in $\mathcal{P}_1$.
- $s = 51$: The product of $x = 17$ and $y = 34$ equals $2 \cdot 17^2$, and is not in $\mathcal{P}_1$.
- $8 \leq s \leq 54$, and s even: Then s can be written as the sum of two distinct, odd primes x and y; hence the corresponding product xy is not in $\mathcal{P}_1$.
- $5 \leq s \leq 53$, and $s = q + 2$ for a prime q: The product of $x = 2$ and $y = q$ is not in $\mathcal{P}_1$.

This leaves us with the ten numbers $11, 17, 23, 27, 29, 35, 37, 41, 47, 53$ as candidates for $\mathcal{S}_2$. The following lines enumerate the compatible products for every candidate:

s=11: 18, 24, 28, 30.

s=17: 30, 42, 52, 60, 66, 70, 72.

s=23: 42, 60, 76, 90, 102, 112, 120, 126, 130, 132.

s=27: 50, 72, 92, 110, 126, 140, 152, 162, 170, 176, 180, 182.

s=29: 54, 78, 100, 120, 138, 154, 168, 180, 190, 198, 204, 208, 210.

s=35: 66, 96, 124, 150, 174, 196, 216, 234, 250, 264, 276, 286, 294, 300, 304, 306.

s=37: 70, 102, 132, 160, 186, 210, 232, 252, 270, 286, 300, 312, 322, 330, 336, 340, 342.

s=41: 78, 114, 148, 180, 210, 238, 264, 288, 310, 330, 348, 364, 378, 390, 400, 408, 414, 418, 420.

s=47: 90, 132, 172, 210, 246, 280, 312, 342, 370, 396, 420, 442, 462, 480, 496, 510, 522, 532, 540, 546, 550, 552.

s=53: 102, 150, 196, 240, 282, 322, 360, 396, 430, 462, 492, 520, 546, 570, 592, 612, 630, 646, 660, 672, 682, 690, 696, 700, 702.

Since all listed products are in $\mathcal{P}_1$, we conclude that set $\mathcal{S}_2$ consists of 11, 17, 23, 27, 29, 35, 37, 41, 47, 53.

We turn to set $\mathcal{P}_3$. A product p is in $\mathcal{P}_3$, if and only if it is compatible with precisely one of the sums in $\mathcal{S}_2$; this means that p shows up in exactly one of the ten enumerations listed above. For instance, the three products 18, 24, 28 only show up for $s = 11$, and hence are contained in $\mathcal{P}_3$. The product 30 shows up once for $s = 11$ and once for $s = 17$, and hence is not in $\mathcal{P}_3$. Here is a cleaned-up version of the above enumerations, that only lists the values in $\mathcal{P}_3$:

s=11: 18, 24, 28.

s=17: 52.

s=23: 76, 112, 130.

s=27: 50, 92, 110, 140, 152, 162, 170, 176, 182.

s=29: 54, 100, 138, 154, 168, 190, 198, 204, 208.

s=35: 96, 124, 174, 216, 234, 250, 276, 294, 304, 306.

s=37: 160, 186, 232, 252, 270, 336, 340.

s=41: 114, 148, 238, 288, 310, 348, 364, 378, 390, 400, 408, 414, 418.

s=47: 172, 246, 280, 370, 442, 480, 496, 510, 522, 532, 540, 550, 552.

s=53: 240, 282, 360, 430, 492, 520, 570, 592, 612, 630, 646, 660, 672, 682, 690, 696, 700, 702.

Finally, we derive $\mathcal{S}_4 = \{17\}$, since the set $\mathcal{P}_3$ contains two or more compatible products for each sum $s \in \mathcal{S}_2$, except for $s = 17$. Hence, $s = 17$ and $p = 52$ with $x = 4$ and $y = 13$ form the unique solution to the classical Freudenthal problem.

5 Stable solutions and phantom solutions for m=2

Martin Gardner [4] attempted to simplify the classical Freudenthal problem for his Scientific American column: He reduced the feasible region to the smaller region $2 \leq x, y \leq 20$, which is easier to handle but still safely contains the numbers 4 and 13 of the supposed solution. This simplification turned out to be fatal, and hundreds of readers pointed out that Gardner's modified problem has no solution at all. In this section, we will discuss problem Freudenthal$^{\neq}(2, M)$ under varying feasible regions, when the bound M grows and tends to infinity.

We will write $\mathcal{P}_1(M), \mathcal{S}_2(M), \mathcal{P}_3(M), \mathcal{S}_4(M)$ to stress that these concepts now depend on M (whereas $m = 2$ is fixed). For a product p, we denote by $\mathcal{M}_1(p)$ respectively $\mathcal{M}_3(p)$ the set of all bounds M with $p \in \mathcal{P}_1(M)$ respectively $p \in \mathcal{P}_3(M)$. Similarly, for a sum s, we denote by $\mathcal{M}_2(s)$ respectively $\mathcal{M}_4(s)$ the set of all bounds M with $s \in \mathcal{S}_2(M)$ respectively $s \in \mathcal{S}_4(M)$. An *interval* $[\ell, r]$ consists of all integers M with $\ell \leq M \leq r$, and a *half-line* $[\ell, \infty]$ of all M with $\ell \leq M$.

Theorem 1. *For any sum s and any product p, the following holds true.*

(a) $\mathcal{M}_1(p)$ *is either empty or a half-line.*

(b) $\mathcal{M}_2(s)$ *is either empty or a half-line.*

(c) $\mathcal{M}_3(p)$ *is either empty or a half-line or an interval.*

(d) $\mathcal{M}_4(s)$ *is either empty or a half-line or an interval.*

Proof. Throughout we will ignore the trivial cases where the considered set is empty.

(a) A product p is in $\mathcal{P}_1(M)$, if and only if it has at least two distinct factorizations under the bound M. The claim now follows from $Z^{\neq}(2, M) \subseteq Z^{\neq}(2, M+1)$.

(b) A sum s lies in $\mathcal{S}_2(M)$, if and only if all compatible products $x(s-x)$ are in $\mathcal{P}_1(M)$. By (a), this is the case if and only if M lies in the intersection of the corresponding half-lines $\mathcal{M}_1(x(s-x))$. This intersection is again a half-line.

(c) A product p lies in $\mathcal{P}_3(M)$, if and only if exactly one of its compatible sums $x + p/x$ lies in $\mathcal{S}_2(M)$. By (b), this is the case if and only if M lies in exactly one of the half-lines $\mathcal{M}_2(x + p/x)$, say in the half-line corresponding to sum $s(p)$. If there are no other half-lines involved, then $\mathcal{M}_3(p)$ coincides with the half-line $\mathcal{M}_2(s(p))$. If there are other half-lines involved, then $\mathcal{M}_3(p)$ is the interval that goes from the endpoint of $\mathcal{M}_2(s(p))$ to the leftmost endpoint of the remaining half-lines. Note that in either case the left endpoint of $\mathcal{M}_3(p)$ coincides with the left endpoint of $\mathcal{M}_2(s(p))$.

(d) Let s be an arbitrary sum. Assume that the products p_a and p_b both are compatible with s, and that there exist two values $M_a, M_b \in \mathcal{M}_2(s)$ such that $p_a \in \mathcal{P}_3(M_a)$ and $p_b \in \mathcal{P}_3(M_b)$. Then the discussion under (c) yields that $s(p_a) = s(p_b) = s$, and that furthermore the left endpoints of $\mathcal{M}_3(p_a)$ and $\mathcal{M}_3(p_b)$ both coincide with the left endpoint of $\mathcal{M}_2(s)$.

A sum s is in $\mathcal{S}_4(M)$, if and only if exactly one of its compatible products $x(s-x)$ lies in $\mathcal{P}_3(M)$. By (c), this is the case if and only if M lies in exactly one of the corresponding half-lines or intervals. By the above discussion, the left endpoints of all these half-lines and intervals coincide with the left endpoint of $\mathcal{M}_2(s)$. Then $\mathcal{M}_4(s)$ is the region covered by exactly one of these half-lines and intervals, and is again a half-line or an interval (or is empty). ∎

For a pair (x, y), we denote by $\mathcal{M}(x, y)$ the set of all integers M for which (x, y) is a solution to FREUDENTHAL$^{\neq}(2, M)$. Theorem 1 yields that $\mathcal{M}(x, y)$ is either a half-line or an interval. We call (x, y) a *stable* solution, if $\mathcal{M}(x, y)$ is a half-line, and we call it a *phantom* solution, if $\mathcal{M}(x, y)$ is an interval. For instance, the pair $(67, 82)$ is a phantom solution that is only active for the range $4.721 \leq M \leq 5.485$.

Theorem 2. *The pair* $(4, 13)$ *forms a stable solution for* FREUDENTHAL$^{\neq}(2, *)$. *The set* $\mathcal{M}(4, 13)$ *consists of all* $M \geq 65$.

Proof. First, we discuss the cases with $M \geq 65$. It is easily verified that the six sums 11, 17, 23, 27, 35, 37 are contained in $\mathcal{S}_2(65)$. Theorem 1.(b) implies that these six sums are also contained in all sets $\mathcal{S}_2(M)$ with $M \geq 65$. As a consequence, the set $\mathcal{P}_3(M)$ does not contain any of the following six products: $30 = 5 \cdot 6 = 2 \cdot 15$; $42 = 2 \cdot 21 = 3 \cdot 14$; $60 = 3 \cdot 20 = 4 \cdot 15$; $66 = 2 \cdot 33 = 6 \cdot 11$; $70 = 2 \cdot 35 = 7 \cdot 10$; and $72 = 3 \cdot 24 = 8 \cdot 9$. On the other hand the product $52 = 4 \cdot 13 = 2 \cdot 26$ lies in $\mathcal{P}_3(M)$, since $17 \in \mathcal{S}_2(M)$ and $28 \notin \mathcal{S}_2(M)$. We now derive $17 \in \mathcal{S}_4(M)$ from this: The sum 17 can be written as $2 + 15$, $3 + 14$, $4+13$, $5+12$, $6+11$, $7+10$, and $8+9$ with corresponding products 30, 42, 52, 60, 66, 70, and 72. Since exactly one of these products lies in $\mathcal{P}_3(M)$, the pair $(4, 13)$ indeed forms a solution for $M \geq 65$. Next, we discuss cases $M \leq 64$. We claim that neither 19 nor 37 is in $\mathcal{S}_2(M)$:

- $2 \cdot 17 \notin \mathcal{P}_1(M)$ implies $19 = 2 + 17 \notin \mathcal{S}_2(M)$.
- $186 \notin \mathcal{P}_1(M)$, since only $186 = 6 \cdot 31$ can be a legal factorization for $M \leq 64$. (In particular the factorization $186 = 3 \cdot 62$ with sum $3 + 62 > M$ is not legal.) Then $186 = 6 \cdot 31 \notin \mathcal{P}_1(M)$ implies $6 + 31 = 37 \notin \mathcal{S}_2(M)$.

Now suppose for the sake of contradiction that the pair $(4, 13)$ forms a solution. Then $17 \in \mathcal{S}_2(M)$ and $52 \in \mathcal{P}_3(M)$. Since the factorizations of 70 are $2 \cdot 35$, $5 \cdot 14$, and $7 \cdot 10$, and since exactly one of the corresponding sums 37, 19, 17 lies in $\mathcal{S}_2(M)$, we get $70 \in \mathcal{P}_3(M)$. Since $\mathcal{P}_3(M)$ contains two products $52 = 4 \cdot 13$ and $70 = 7 \cdot 10$ compatible with the sum 17, we get $17 \notin \mathcal{S}_4(M)$. Hence, the pair $(4, 13)$ cannot be a solution for $M \leq 64$. ■

The pair $(4, 13)$ is actually the *unique* solution of FREUDENTHAL$^{\neq}(2, M)$ for $65 \leq M \leq 1.684$. For $M \leq 64$ there are no solutions, and for $M \geq 1.685$ the pair $(4, 61)$ forms a second solution. Martin Gardner conjectured in private correspondence with John Kiltinen and Peter Young (mentioned in the introduction of [5]) that the number of solution pairs for FREUDENTHAL$^{\neq}(2, *)$ should be infinite. To the best of our knowledge, this conjecture is still open. We propose the following slight strengthening.

Conjecture 3. FREUDENTHAL$^{\neq}(2, *)$ *has infinitely many stable solutions.*

Many stable solutions for FREUDENTHAL$^{\neq}(2, *)$ contain a power of 2, but not all of them do: The pair $(201, 556)$ is a stable solution that is active for all $M \geq 966.293$. Section 8 provides additional information on stable solutions for FREUDENTHAL$^{\neq}(2, *)$.

6 A meta-variant of Freudenthal

In September 2000, Clive Tooth created a kind of Meta-Freudenthal problem, and posed it to the readers of the newsgroup `sci.math` on the Usenet. We present it in a slightly modified form that is built around the solutions of problem FREUDENTHAL$^{=}(2, 5.000)$.

> The teacher says to Peter and Sam: I have secretly chosen two integers x and y with $2 \leq x \leq y$ and $x+y \leq 5.000$. I have told their sum $s = x+y$ only to Sam and their product $p = xy$ only to Peter.

1. Peter says: I don't know the numbers x and y.
2. Sam replies: I already knew you didn't know.
3. Peter says: Oh, then I do know the numbers x and y.
4. Sam says: Oh, then I also know them.

Up to this point, John has listened quietly to the conversation.

5. John complains: But I still don't know the numbers x and y.
6. Sam replies: But if we told you the value x, then you could determine y.
7. John says: Oh, then I do know the numbers x and y.

Determine x and y!

Denniston's algorithm for $\text{FREUDENTHAL}^{=}(2, 5.000)$ yields ten possible solution pairs that agree with the first four statements of the conversation; these ten pairs are listed in Table 4. Since the values $x = 4$, $x = 16$, $x = 32$, and $x = 64$ do not uniquely determine the corresponding y, we conclude (together with John) that the answer must be the (phantom) solution $x = 67$ and $y = 82$.

	1	2	3	4	5	6	7	8	9	10
x	4	4	4	16	16	32	32	64	64	67
y	13	61	229	73	111	131	311	73	309	82
s	17	65	233	89	127	163	343	137	373	149
p	52	244	916	1.168	1.776	4.192	10.976	4.672	19.776	5.494

Table 4: Ten intermediate solutions for the Meta-Freudenthal problem.

If the meta-variant is built around problem $\text{FREUDENTHAL}^{\neq}(2, 5.000)$ instead of $\text{FREUDENTHAL}^{=}(2, 5.000)$, then $x = 67$ and $y = 82$ remains the unique answer. However, the line of argument changes slightly, since $\text{FREUDENTHAL}^{\neq}(2, 5.000)$ only possesses five feasible solutions, which are the first, second, fourth, fifth, and tenth solution in Table 4.

7 A Mediterranean variant of Freudenthal

The *Mediterranean Mathematical Olympiad* (MedMO) is an annual mathematical competition for high-school students from all countries which either have a Mediterranean coast or are adjacent to a country with a Mediterranean coast. Here is a slightly adapted version of the first problem posed at MedMO'2005:

> The teacher says to Peter and Sam: I have secretly chosen two positive integers x and y with $x \leq y$. I have told their sum $s = x + y$ to Sam and their product $p = xy$ to Peter.
>
> 1. Sam says: You are not able to determine s.
> 2. Peter says: Aha. But now I know that $s = 136$.
>
> Determine x and y!

Note that Sam's statement #1 summarizes and contracts the first and second statement in the Freudenthal problem $\text{FREUDENTHAL}^{=}(1, *)$: Peter is not able to work out the numbers x and y from the product p, and Sam is aware of this fact. We will demonstrate below that at the end of the conversation, Peter and Sam both know the numbers x and y. Hence, the above conversation is equivalent to the standard Freudenthal problem, except that Peter explicitly reveals the sum $s = 136$ to the reader.

If we replace the value 136 in the conversation by an arbitrary positive integer z, then we arrive at the Mediterranean Freudenthal problem $\text{MED}(z)$. In this section, we will fully analyze and understand all these Mediterranean problems. Some standard definitions: A divisor d of a positive integer z is *proper*, if $1 < d < z$. An integer $z \geq 2$ is *composite*, if it has some proper divisor. Our analysis is structured into three observations.

First: Consider the moment just before statement #1. If the product p is prime, then Peter would already know at that moment that $x = 1$ and that $y = p$. If the product p is composite, then Peter cannot distinguish between the case where $x = 1$ and the case where x is the smallest proper divisor of p. This yields that p must be composite.

Second: We conclude that statement #1 is equivalent to the following: For all positive integers x and y with $x + y = s$, the product xy is composite. And it is not hard to see that this statement simply boils down to: The number $s - 1$ is composite.

Third: Let $1 = d_1 < d_2 < \cdots < d_k$ be an enumeration of the divisors of p that are less or equal to $\sqrt{p}$. Then at the time point just before statement #2, the values $s_i := d_i + p/d_i$ ($i = 1, \ldots, k$) are Peter's current candidate values for the sum s. The Mediterranean problem has a solution, if and only if Peter can exclude all these candidates except one. And Peter can exclude the candidate s_i, if and only if $s_i - 1$ is prime. Consequently, with a single exception all the values $s_i - 1$ must be prime. And we already have identified this single exception: Since $d_1 = 1$, the value $s_1 - 1 = d_1 + p/d_1 - 1$ equals p, and we observed above that p is composite. Hence, $p = s - 1$, $x = 1$, and $y = p$.

We summarize the above observations in the following theorem.

Theorem 4. *Let $z \geq 2$ be an integer. The Mediterranean problem* MED(z) *is well-posed and possesses a unique solution, if and only if z is a so-called Mediterranean number, that is, a number that satisfies the following properties:*

- $z - 1$ *is composite*
- $d + (z - 1)/d - 1$ *is prime, for any proper divisor d of $z - 1$*

Furthermore, the unique solution in this case is $x = 1$ and $y = z - 1$, and the corresponding product is $p = z - 1$.

Let us quickly verify this theorem for problem MED(136), the well-posed problem from MedMO'2005: Clearly 135 is composite. The factorizations of 135 into two proper factors are $3 \cdot 45$, $5 \cdot 27$, and $9 \cdot 15$. And indeed, the three corresponding candidate sums $3 + 45 - 1 = 47$, $5 + 27 - 1 = 31$, and $9 + 15 - 1 = 23$ all are prime. Therefore 135 is a Mediterranean number, and the unique answer for MED(136) is $x = 1$ and $y = 135$.

The reader may want to check that 5, 9, 10, 16, 28, 33, 34, 36, 46, and 50 are the first ten Mediterranean numbers. Also 666 (the number of the beast) is a Mediterranean number. Altogether, 39.821 of the integers below 1.000.000 are Mediterranean numbers. We leave the following challenge to the reader: Is there a polynomial time algorithm for deciding whether a given number z is Mediterranean?

8 More stable solutions and phantom solutions

This section continues the discussion in Section 5. We investigate the solution sets for FREUDENTHAL$^{\neq}(m, M)$ and FREUDENTHAL$^{=}(m, M)$ as M grows while m is fixed. Theorem 1 easily generalizes to $m \geq 1$, and thus yields the classification into stable solutions and phantom solutions for any fixed $m \geq 1$.

Let us start with the case $m = 1$. The stable solutions for problem FREUDENTHAL$^{=}(1, *)$ are easy to describe, since they are closely related to Theorem 4: A pair forms a stable solution, if and only if it is of the form $(1, z-1)$ where z is a Mediterranean number. The stable solutions for problem FREUDENTHAL$^{\neq}(1, *)$ can be characterized in a similar fashion: A pair is a stable solution, if and only if it is of the form $(1, z-1)$ where z satisfies the following two almost-Mediterranean properties:

- $z - 1$ is neither prime, nor the square of a prime
- $d + (z - 1)/d - 1$ is prime or the square of a prime, for any proper divisor d of $z - 1$ with $d^2 \neq z - 1$

		FREUDENTHAL$^=(2, M)$		FREUDENTHAL$^{\neq}(2, M)$	
x	y	$x + y \in \mathcal{S}_2$	solution	$x + y \in \mathcal{S}_2$	solution
4	13	$28 \le M$	$65 \le M$	$28 \le M$	$65 \le M$
4	61	$124 \le M$	$869 \le M$	$173 \le M$	$1.685 \le M$
32	131	$317 \le M$	$1.505 \le M$	$317 \le M$	$9.413 \le M$
16	73	$169 \le M$	$1.970 \le M$	$169 \le M$	$1.970 \le M$
16	111	$233 \le M$	$2.522 \le M$	$233 \le M$	$2.522 \le M$
32	311	$677 \le M$	$3.832 \le M$	$677 \le M$	$6.245 \le M$
64	73	$265 \le M$	$4.037 \le M$	$265 \le M$	$6.245 \le M$
4	229	$460 \le M$	$4.628 \le M$	$460 \le M$	$6.893 \le M$
8	239	$485 \le M$	$7.787 \le M$	$485 \le M$	$72.365 \le M$
4	181	$364 \le M$	$7.898 \le M$	$1.373 \le M$	$237.173 \le M$
16	163	$349 \le M$	$7.940 \le M$	$349 \le M$	$7.940 \le M$
64	127	$367 \le M$	$9.104 \le M$	$367 \le M$	$9.104 \le M$

		FREUDENTHAL$^=(3, M)$		FREUDENTHAL$^{\neq}(3, M)$	
x	y	$x + y \in \mathcal{S}_2$	solution	$x + y \in \mathcal{S}_2$	solution
13	16	$49 \le M$	$98 \le M$	$49 \le M$	$125 \le M$
16	73	$169 \le M$	$961 \le M$	$169 \le M$	$9.413 \le M$
64	127	$367 \le M$	$1.783 \le M$	$367 \le M$	$5.045 \le M$
16	133	$283 \le M$	$2.767 \le M$	$283 \le M$	$6.893 \le M$
16	163	$349 \le M$	$5.300 \le M$	$349 \le M$	$5.300 \le M$
16	223	$469 \le M$	$5.761 \le M$	$469 \le M$	$332.933 \le M$
64	367	$847 \le M$	$5.821 \le M$	$847 \le M$	$18.773 \le M$
16	193	$403 \le M$	$7.229 \le M$	$403 \le M$	$7.229 \le M$
64	457	$1.024 \le M$	$9.349 \le M$	$1.024 \le M$	$36.485 \le M$

Table 5: Some stable solutions for $m = 2$ and $m = 3$.

Since the arguments are similar to those in Section 7, we leave all details to the reader. The smallest stable solution for FREUDENTHAL$^=(1, *)$ is $(1, 4)$, which is active for all $M \ge 11$. The smallest stable solution for FREUDENTHAL$^{\neq}(1, *)$ is $(1, 6)$, which is active for all $M \ge 23$. There are plenty of phantom solutions for FREUDENTHAL$^=(1, *)$ and FREUDENTHAL$^{\neq}(1, *)$, and they do not seem to have interesting properties. We only mention that the phantom solution $(3, 4)$ for FREUDENTHAL$^=(1, *)$ is particularly simple and can be verified by hand; it is active for the range $16 \le M \le 22$.

Now let us turn to $m = 2$ and $m = 3$. The left half of Table 5 lists all stable so-

x	y	FREUDENTHAL$^=(2, M)$ active in the interval	FREUDENTHAL$^{\neq}(2, M)$ active in the interval
64	309	$4.625 \leq M \leq 13.168$	$187.493 \leq M \leq 1.739.764$
67	82	$4.721 \leq M \leq 5.485$	$4.721 \leq M \leq 5.485$
139	192	$10.975 \leq M \leq 17.788$	——
149	188	$12.353 \leq M \leq 14.004$	——
83	248	——	$17.789 \leq M \leq 19.324$
96	241	——	$16.133 \leq M \leq 22.804$

Table 6: Some phantom solutions for $m = 2$.

lutions for FREUDENTHAL$^=(2, *)$ and FREUDENTHAL$^=(3, *)$ that enter the scene before $M = 10.000$. The right half of the table lists the corresponding data for problems FREUDENTHAL$^{\neq}(2, *)$ and FREUDENTHAL$^{\neq}(3, *)$. Note that the stable solutions in both halves of the table are the same. This is not just a lucky coincidence:

Theorem 5. *Assume that the following modification of Goldbach's conjecture holds true: Every even number $s \geq 8$ can be written as the sum of two distinct primes. Then for $m = 2$ and $m = 3$, the stable solutions of* FREUDENTHAL$^=(m, *)$ *coincide with the stable solutions of* FREUDENTHAL$^{\neq}(m, *)$.

Proof. Since we deal with stable solutions, the upper bounds M do not play any role and will be ignored throughout. We observe that $\mathcal{S}_2$ only contains odd numbers: The sums $s = 4$ and $s = 6$ obviously cannot be in $\mathcal{S}_2$. Modified Goldbach yields that every even sum $s \geq 8$ is compatible with the product of two distinct primes, and hence not in $\mathcal{S}_2$.

Now consider a product of the form q^4, where q is prime. This product has at most one factorization $q^4 = xy$ with $(x, y) \in Z^{\neq}(m, *)$, but may have two distinct factorizations with $(x, y) \in Z^=(m, *)$. The main difference between the two variants (without upper bound M) is that these products q^4 will show up in the set $\mathcal{P}_1$ for one variant, but not in $\mathcal{P}_1$ for the other variant. However this will not affect the sets $\mathcal{S}_2$, since $\mathcal{S}_2$ only contains odd numbers, whereas the factorizations of q^4 concern the even numbers $q + q^3$ and $2q^2$. ■

We have checked all pairs (x, y) with $x + y \leq 50.000$ with the help of a computer program. Among these pairs there are 1.796 stable solutions and 689 phantom solutions for FREUDENTHAL$^=(2, *)$ and FREUDENTHAL$^{\neq}(2, *)$, and there are 804 stable solutions and 288 phantom solutions for FREUDENTHAL$^=(3, *)$ and FREUDENTHAL$^{\neq}(3, *)$. Some of the phantom solutions for $m = 2$ are listed in Table 6. The smallest phantom solution for $m = 3$ is $(123, 128)$; it is active in the interval $[2.870, 10.480]$ for FREUDENTHAL$^=(3, *)$ and active in the interval $[6.893, 10.480]$ for FREUDENTHAL$^{\neq}(3, *)$.

The behavior of the cases with $m \geq 4$ is not understood. We are not aware of *any* solution for *any* of these problems. In particular, we have not found any solutions (x, y) with $x + y \leq 50.000$.

Conjecture 6. *For $m \geq 4$, problems* FREUDENTHAL$^{=}(m, *)$ *and* FREUDENTHAL$^{\neq}(m, *)$ *do not have any solutions.*

9 Yet another Freudenthal problem

All the Freudenthal problems discussed in this article contained a statement of the type *"I already knew that you didn't know"*, which in some sense is their common theme. Here is a final puzzle of this type:

> The teacher says to Peter and Sam: I have secretly chosen two integers x and y with $1 \leq x \leq y$. I have told their sum $s = x + y$ only to Sam and their product $p = xy$ only to Peter.
>
> 1. Peter says: I don't know the numbers x and y.
> 2. Sam says: I already knew that. The sum is less than 14.
> 3. Peter says: I already knew that. But now I know the numbers x and y.
> 4. Sam says: Oh, then I also know them.
>
> Determine x and y!

It is not difficult to show that $x = 2$ and $y = 9$, and we leave this to the reader.

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