On the geometry of $k$-transvection groups

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1. Introduction

More than 20 years ago, B. Fischer [5,6] started his pioneering work on groups generated by a conjugacy class of 3-transpositions, i.e., a conjugacy class of involutions $D$, such that for all $d, e \in D$, we have that $[d, e] = 1$ or $\langle d, e \rangle \simeq SL_2(2)$. Fischer classified all finite groups $G$ generated by a class of 3-transpositions under the additional assumptions that $G'' = G'$ and $O_2(G) = O_3(G) = Z(G) = 1$.

Fischer’s work has been generalized in two directions. On one hand, J.I. Hall and the author [4] have obtained an almost complete classification of all centerfree 3-transposition groups allowing infinite groups and nontrivial normal 2- or 3-groups. See also [8,9]. On the other hand, Aschbacher [1] and more recently Timmesfeld [7] have extended Fischer’s work to groups generated by a class of $k$-transvection groups, for arbitrary fields $k$, where $k$-transvection groups are defined as follows.

Suppose $k$ is a field, $G$ a group and $\Sigma$ a conjugacy class of abelian subgroups of $G$ such that

1. $G = \langle \Sigma \rangle$;
2. for all $A, B \in \Sigma$ we either have that $[A, B] = 1$, or $A$ and $B$ are full unipotent subgroups of $\langle A, B \rangle \simeq \langle P \rangle SL_2(k)$;

then $\Sigma$ is called a class of $k$-transvection (sub)groups in $G$.

Timmesfeld considers groups $G$ generated by a class of $k$-transvection subgroups having no nontrivial nilpotent normal subgroups and satisfying some finiteness condition, the ascending chain condition, i.e., all ascending chains of $\Sigma$-subgroups are of finite length. For fields with more than 3 elements, he obtains an almost complete classification by showing that the group under consideration is contained in the automorphism group of a polar space and that
the elements of the abstract \( k \)-transvection subgroups induce transvections (or elations) on these polar spaces.

Motivated by the successful use of the geometry of 3-transpositions, i.e., the Fischer spaces, in the joined work with J.I. Hall on groups generated by 3-transpositions, and by Timmesfeld’s work, we started the investigation of the geometry of \( k \)-transvection groups, in the hope to obtain a more elementary proof of Timmesfeld’s results and to remove his finiteness conditions. Here we report on those attempts.

In the next section we show how to obtain a geometry from a class of \( k \)-transvection groups and derive some properties of such a geometry. In Section 3 we discuss some of the results we have obtained on geometries and groups generated by a class of \( k \)-transvection subgroups. These results and their proofs can be found in [2,3].

2. The geometry of \( k \)-transvection groups

Suppose \( k \) is a field and \( \Sigma \) is a class of \( k \)-transvection subgroups generating the group \( G \). Then we have the following.

**Lemma 2.1** Let \( A, B \) be two noncommuting elements in \( \Sigma \) and \( X = \langle A, B \rangle \), then the following hold:

1. for all distinct \( C, D \in \Sigma \) that are contained in \( X \) we have that \( X = \langle C, D \rangle \);
2. \( X \) contains \( |k| + 1 \) elements of \( \Sigma \);
3. \( A \) acts regularly on the set of elements of \( \Sigma \) in \( X \) different from \( A \);
4. for all \( a_1 \in A^\perp \) and \( b_1, b_2 \in B^\perp \) we have \( B^{a_1 b_2} = A \) or there are \( a_2, a_3 \in A \) and \( b_3 \in B \) with \( a_3 b_1 a_2 = b_3 a_1 b_2 \).

**Proof.** These are properties of \( SL_2(k) \), and can be computed easily. \( \square \)

Let \( L(\Sigma) \) be the set of sets \( \{ C \in \Sigma \mid C \leq \langle A, B \rangle \} \) where \( A \) and \( B \) are noncommuting elements of \( \Sigma \). Then by (1) of the above lemma we have that \( \Pi = (\Sigma, L(\Sigma)) \) is a partial linear space, called the geometry of \( k \)-transvection subgroups. As \( \Sigma \) is a conjugacy class in \( G = \langle \Sigma \rangle \) we have that \( \Pi \) is connected. The elements of \( \Sigma \) will be called points and those of \( L(\Sigma) \) will be called lines of \( \Pi \). From 2.1.2 it follows that all lines of \( \Pi \) contain \( |k| + 1 \) points.

If \( A \) and \( B \) are two points in \( \Sigma \), then by \( A \perp B \) we denote that \( A \) and \( B \) commute. By \( A^\perp \) we denote the set of all points in \( \Sigma \) commuting with \( A \). If \( A \) and \( B \) do not commute, then by \( AB \) we denote the unique line containing \( A \) and \( B \). A triangle is a triple \( (A, B, C) \) of distinct points with \( A \) and \( B \), and \( B \) and \( C \) collinear, but \( A \perp C \). The following proposition is a consequence of 2.1.4.
Proposition 2.2 Any triangle in $\Pi$ generates a subspace isomorphic to a dual affine plane, i.e., a projective plane from which a point and all the lines through that point are removed.

Sketch of the Proof. The main step in the proof of the proposition is to show that Pasch’s axiom holds in the subspace generated by $A, B$ and $C$. For that purpose let $F$ be a point on $AB$, and $D, E$ distinct points on $BC$ all distinct from $A, B$ and $C$. We show that $AD$ meets $EF$.

As follows from 2.1.3, there are $a_1 \in A^\perp$ and $b_1, b_2 \in B^\perp$ with $D = C^{b_1}$, $E = C^{b_2}$ and $F = B^{a_1b_2}$. By 2.1.4 there are $a_2, a_3 \in A$ and $b_3 \in B$ with $a_3b_1a_2 = b_3a_1b_2$, and hence $AD \ni D = C^{a_3b_1a_2} = (C^{b_3})^{a_1b_2} = EF$.

In a dual affine plane every point is on a unique maximal coclique of the collinearity graph of the coclique. These cocliques are called the transversal cocliques of the dual affine plane.

Let $(A, B, C)$ be a triangle. Using some knowledge of the action of $\langle A, B \rangle$ on the dual affine plane generated by $A, B$ and $C$, we are able to prove the following.

Lemma 2.3 Suppose $k$ contains at least 4 elements.

1. If $A$ is a point and $T$ a transversal coclique of $\Pi$ with $|T \cap A^\perp| \geq 2$, then $T \subseteq A^\perp$;
2. let $A$ and $B$ be points in $\Sigma$ with $A^\perp \subseteq B^\perp$, then $A^\perp = B^\perp$.

Thus we have obtained:

Proposition 2.4 Let $\Sigma$ be a class of $k$-transvection groups. If $k$ contains at least 4 elements, then $(\Sigma, L(\Sigma))$ satisfies the following:

1. if $p$ and $q$ are two points in $\Sigma$ with $p^\perp \subseteq q^\perp$ then $p^\perp = q^\perp$;
2. any triangle generates a subspace isomorphic to a dual affine plane;
3. if $p$ is a point and $T$ a transversal coclique with $|T \cap p^\perp| \geq 2$, then $T \subseteq p^\perp$.
4. all lines contain at least 4 (in fact at least 5) points.

3. The results

Let $\Pi = (P, L)$ be a connected partial linear space having the properties of 2.4. For points $p$ and $q$ we write $p \ast q$ if and only if $p^\perp = q^\perp$. The relation $\ast$ is an equivalence relation, and we can consider the quotient geometry $\overline{\Pi} = (\overline{P}, \overline{L})$ of $\Pi$ with respect to this relation. Here $\overline{P}$ is the set of $\ast$-equivalence classes of $P$, and $\overline{L}$ is the set of subsets of $\overline{P}$ of the form $\{\overline{p} \in \overline{P} \mid \overline{p}$ meets $l$ nontrivially$\}$,
where \( l \in L \). Then \( \Pi \) is a geometry also having the properties of 2.4, however, with the stronger property that for all points \( p \) and \( q \) we have that \( p^\perp \subseteq q^\perp \) implies \( p = q \).

Now assume that \( \Pi = \Xi \). For any two noncollinear points \( p \) and \( q \) define the singular line through \( p \) and \( q \) to be the set

\[
(pq)^\perp = \{ r \in P \mid r^\perp \supsetneq p^\perp \cap q^\perp \}.
\]

It follows from 2.3 that any transversal coclique on \( p \) and \( q \) is contained in the singular line on \( p \) and \( q \). Thus all singular lines contain at least 3 points. In fact we are able to prove the following.

**Theorem 3.1** If \( \Pi \) contains noncollinear points, then \( P \) together with its singular lines is a disjoint union of nondegenerate polar spaces.

**Sketch of the Proof.** To prove the theorem we have to check the Buekenhout-Shult axiom. For that purpose we have to consider a singular line \( l \) and a point \( p \) not on \( l \). If \( p^\perp \cap l \neq \emptyset \), then by the definition of singular lines it contains a unique point or equals \( l \). Hence, we only have to show that this intersection is never empty.

Let \( q \) and \( r \) be two points on \( l \), not in \( p^\perp \). Then by some connectivity properties of \( \Pi \) we are able to find a point \( s \) in \( p^\perp \cap q^\perp \), but collinear to \( r \). Inside the subspace of \( \Pi \) generated by \( p \), \( q \), \( r \) and \( s \), we are able to find a transversal coclique on \( q \) and \( r \) and thus contained in \( l \), meeting \( p^\perp \), which proves the theorem. \( \Box \)

**Corollary 3.2** Let \( k \) be a field and \( G \) a group generated by a conjugacy class \( \Sigma \) of \( k \)-transvection subgroups. If \( k \) contains at least 4 elements, \( G \) has no nontrivial nilpotent normal subgroups and \( \Sigma \) contains two commuting elements, then \( G \) is contained in the automorphism group of a nondegenerate polar space, on which the elements of subgroups in \( \Sigma \) act as transvections.

**Sketch of the Proof.** Consider the geometry of \( k \)-transvection groups \( \Pi = (\Sigma, L(\Sigma)) \). Then \( G \) acts by conjugation on this geometry. As the center of \( G \) can be assumed to be trivial, this action is faithful. By the results of the previous section, there are two obvious sets of imprimitivity for this action, the \( \perp \)-connected components of the graph \( (\Sigma, \perp) \), and the set of *-equivalence classes. If one of these sets of imprimitivity is nontrivial, then let \( N \) be the normal subgroup of \( G \) consisting of all elements in \( G \) acting trivially on this particular set of imprimitivity. We show that \([N, G] \) is a nontrivial nilpotent normal subgroup of \( G \). Hence we can apply the above theorem, and the corollary follows easily. \( \Box \)

**References**


