The geometry of $k$-transvection groups

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To Professor Bernd Fischer
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Abstract

Let $k$ be a (commutative) field and $G$ a group, then a conjugacy class of Abelian subgroups of $G$ is called a class of $k$-transvection subgroups in $G$ if and only if it generates $G$ and any two elements of the class either commute or are full unipotent subgroups of the group they generate and which is isomorphic to (P)SL$_2(k)$.

In this paper we study the geometry of $k$-transvection groups. Given a class of $k$-transvection groups $\Sigma$, we consider a partial linear space whose points are the elements of $\Sigma$, and whose lines correspond to the groups generated by two non-commuting elements from $\Sigma$. We derive several properties of this partial linear space. These properties are used to give a characterization of the geometries of $k$-transvection groups and provide a classification of groups generated by $k$-transvection subgroups.

1 Introduction

Let $G$ be a group and $k$ a field. Then a conjugacy class $\Sigma$ of Abelian subgroups of $G$ is called a class of $k$-transvection (sub)groups of $G$, if the following hold:

(a) $G = \langle \Sigma \rangle$;

(b) for all $A, B \in \Sigma$ we have $[A, B] = 1$, notation $A \perp B$, or $A$ and $B$ are full unipotent subgroups of the group $\langle A, B \rangle$ which is isomorphic to (P)SL$_2(k)$. 

1
Groups generated by a class of $k$-transvection subgroups have been studied by various people. The first to consider $k$-transvection groups was B. Fischer [13,14]. He looked at the special case of finite groups generated by $k$-transvection subgroups, $k$ being the field of two elements. In this case the non identity elements of the transvection subgroups form a class of 3-transpositions. Fischer classified all finite almost simple groups generated by a class of 3-transpositions, leading him not only to various classical groups over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ but also to the three sporadic groups called after him.

Various people have extended Fischer’s work to larger fields. Aschbacher [1], Aschbacher and M. Hall Jr. [2], and J.I. Hall [17] consider the case where $k$ is a field with more than 2 elements but $G$ still finite. Infinite fields and groups have been considered by Timmesfeld in his work on groups generated by $k$-transvection groups [20] and abstract root subgroups [22] but also by Cuypers [7] and Steinbach [19]. Also Fischer’s results on 3-transposition groups have been extended to include infinite groups. In particular, Jonathan Hall and the present author obtained an almost complete classification of all (finite and infinite) center free groups generated by 3-transpositions; see [9].

An important tool in the work of Cuypers and Hall is the Fischer space associated to a class of 3-transpositions. This is the point-line geometry with as points the 3-transpositions and as lines those triples of 3-transpositions contained in a subgroup generated by two non commuting 3-transpositions. These Fischer spaces are partial linear spaces with three points per line such that any two intersecting lines generate a subspace isomorphic to a dual affine plane of order 2 or an affine plane of order 3. It was noticed by Buekenhout [4], that this property characterizes Fischer spaces.

In this paper we consider the geometry of $k$-transvection groups, where $k$ is an arbitrary field. Suppose $\Sigma$ is a class of $k$-transvection subgroups in a group $G = \langle \Sigma \rangle$. In analogy of the Fischer spaces, we define the geometry of $\Sigma$ to be the point-line geometry with as point set $\Sigma$ and as lines the subsets $\{ C \in \Sigma | C \subseteq \langle A, B \rangle \}$, where $A, B$ are non commuting elements from $\Sigma$. This geometry will be denoted by $\Pi(\Sigma)$, its line set by $L(\Sigma)$. We will derive the following.

1.1 Proposition. Suppose $k$ is a field with at least 4 elements and $G$ a group generated by a class $\Sigma$ of $k$-transvection subgroups containing some commuting elements $A$ and $B$ with $C_\Sigma(A) \neq C_\Sigma(B)$.

Then $\Pi := \Pi(\Sigma)$ is a connected partial linear space which satisfies the following.

(a) If $l$ is a line and $x$ a point of $\Pi$ collinear to one but not all points of $l$, then $x$ and $l$ generate a subspace of $\Pi$ isomorphic to a dual affine plane.

(b) If $X$ is a subspace of $\Pi$ isomorphic to a dual affine plane and $p$ a point not collinear with two non-collinear points $q$ and $r$ of $X$, then $p$ is not collinear to all points of $q \perp \cap X$. 

2
(c) If $x$ and $y$ are points with $x^\perp \subseteq y^\perp$ then $x^\perp = y^\perp$; moreover, if $G$ contains no nontrivial nilpotent normal subgroup, then $x^\perp \subseteq y^\perp$ implies $x = y$.

(d) If $\Pi$ contains two points $x \perp y$ with $x^\perp \neq y^\perp$, then $\Pi$ is coconnected (i.e., the graph $(\Sigma, \perp)$ is connected).

(Here $x^\perp$ denotes the set of all points $y$ with $y \perp x$.)

The above result leads us to the following definition.

1.2 Definition. A connected and coconnected partial linear space $\Pi = (P, L)$ satisfying

(a) if $l \in L$ is a line and $x \in P$ a point of $\Pi$ collinear to one but not all points of $L$, then $x$ and $L$ generate a subspace of $\Pi$ isomorphic to a dual affine plane;

(b) if $X$ is a subspace of $\Pi$ isomorphic to a dual affine plane and $p$ a point not collinear with two non-collinear points $q$ and $r$ of $X$, then $p$ is not collinear to all points of $q^\perp \cap X$;

(c) if $x$ and $y$ are points with $x^\perp \subseteq y^\perp$ then $x^\perp = y^\perp$;

is called a transvection geometry.

A partial linear space $S = (P, L)$ is called a polar space if it satisfies the ‘one or all’ axiom:

if $x \in P$ and $l \in L$ then either one or all points of $l$ are collinear to $x$.

The polar graph of a polar space $S$ is the graph whose vertex set is the point set of $S$, two vertices being adjacent if and only if they are collinear. For basic results and terminology on polar spaces the reader is referred to the chapters of Cameron and Cohen in the Handbook of Incidence Geometry [5].

Transvection geometries are closely related to polar spaces as follows from the following result which we have proven in [12]. Before stating it, we introduce a bit of notation.

Suppose $\Pi = (P, L)$ is a transvection geometry. On the point set $P$ of $\Pi$ we can define the relation $\equiv$ by $x \equiv y$ if and only if $x^\perp = y^\perp$. Clearly $\equiv$ is an equivalence relation. The $\equiv$-class of a point $x$ is denoted by $[x]$ and the set of all $\equiv$-classes by $P/ \equiv$. On $P/ \equiv$ we define the relation $\perp$ by $[x] \perp [y]$ if and only if $x \perp y$. (Notice that this is well defined.)
1.3 Theorem ([12]). Let $\Pi = (P, L)$ be a transvection geometry with all lines in $L$ containing at least 4 points. If there are $x, y \in P$ with $x \perp y$ but $x^\perp \neq y^\perp$, then $(P/\equiv, \perp)$ is a non degenerate polar graph of rank at least 2.

As corollary to the above theorem we obtain the following result, also obtained by Timmesfeld [20, 22].

1.4 Theorem. Suppose $k$ is a field with at least 4 elements. Let $G$ be a group generated by a class $\Sigma$ of $k$-transvection subgroups. Suppose that $G$ contains no nontrivial normal nilpotent subgroup and that there are $A \perp B \in \Sigma$ with $C_\Sigma(A) \neq C_\Sigma(B)$. Then $G$ is contained in the automorphism group of a non degenerate polar space $S$ of rank at least 2. The non identity elements from $A \in \Sigma$ act as polar transvections on $S$.

Here a polar transvection of a polar space $S$ is a non trivial automorphism of $S$ fixing a point $p$ of $S$ and all points in $p^\perp$. If $S$ is non degenerate, then the point $p$ is uniquely determined and called the center of the polar transvection.

As in [22], the classification of polar spaces of rank at least 3, [18, 23] and Moufang generalized quadrangles, cf. Tits and Weiss [24], leads to a full classification of the groups appearing in the conclusion of Theorem 1.4; see [20, 22]. We refer the reader to [20, 22] for a description of the groups involved.

The analogue of Buekenhout’s observation that each connected Fischer space can be obtained from a class of 3-transpositions in a group, is not so obvious for transvection geometries. Under some extra conditions, however, we can associate a class of $k$-transvection subgroups to a transvection geometry. Indeed, with the help of Theorem 1.3 and some ideas by Jonathan Hall [16, 17], we can prove the following.

1.5 Theorem. Suppose $\Pi = (P, L)$ is a transvection geometry with at least 4 points per line such that the following hold.

(a) If $l$ is a line and $x$ a point not on $l$ collinear with all points of $l$, then the subspace of $\Pi$ generated by $l$ and $x$ is generated by any three non collinear points in it. Moreover, there exists a point $y$ collinear with $x$ but no point of $l$.

(b) There are three pairwise non collinear points $x_1, x_2, x_3$ with $x_i^\perp \cap x_j^\perp \nsubseteq x_k^\perp$ for $\{i, j, k\} = \{1, 2, 3\}$.

Then there is a field $k$ and a group $G$ generated by a class $\Sigma$ of $k$-transvection subgroups such that $\Pi$ is isomorphic to $\Pi(\Sigma)$.

The above theorem is a step towards a classification of all transvection geometries satisfying the hypothesis of (1.5) as it enables one to us both geometric and group theoretic tools as in [16, 17] (see also [21]).
The geometry of a class of $\mathbb{F}_3$-transvection groups is not necessarily a transvection geometry as it might violate condition (b) of Definition 1.2. Indeed, the class of triflection subgroups (i.e., subgroups generated by reflections of order 3) in unitary groups over the field $\mathbb{F}_4$ forms a class of $\mathbb{F}_3$-transvection groups leading to a geometry violating (1.2)(b). But, this class of examples is the only class not satisfying condition (1.2)(b) as follows by the following result.

1.6 Theorem. Let $G$ be a group generated by a class $\Sigma$ of $\mathbb{F}_3$-transvection subgroups. Suppose $G$ contains no nontrivial normal nilpotent subgroup. Then up to isomorphism we have one of the following.

(a) $\Sigma$ is the class of symplectic transvection subgroups of $PSp(V, f)$ for some non degenerate symplectic $\mathbb{F}_3$-space $(V, f)$;

(b) $\Sigma$ is the class of unitary transvection subgroups of $PU(V, h)$ for some non degenerate unitary $\mathbb{F}_3^2$-space $(V, h)$;

(c) $\Sigma$ is the class of triflection subgroups of $PU(V, h)$ for some non degenerate unitary $\mathbb{F}_2^2$-space $(V, h)$.

In the next section we will derive several properties of the geometry $\Pi(\Sigma)$ of $k$-transvection subgroups. In particular, we show that $\Pi(\Sigma)$ is a transvection geometry, provided the field $k$ contains at least 4 elements. This proves Proposition 1.1 which together with Theorem 1.3 implies Theorem 1.4. In Section 3 we investigate the exceptional case of $\mathbb{F}_3$-transvection groups and provide a proof of Theorem 1.6. The final section is devoted to a proof of Theorem 1.5.

2 The geometry of $k$-transvection groups

In this section we start the investigation of the geometry of $k$-transvection groups. Thus let $k$ be a field and $\Sigma$ be a class of $k$-transvection groups generating the group $G$. We assume $k$ to have at least 3 elements. (Various of the results presented in this section do also hold in the case that $k = \mathbb{F}_2$, but as we do not cover the theory of 3-transposition groups in this paper, we avoid this particular case.)

In the next lemma we collect some elementary properties of $\Sigma$.

2.1 Let $A$ and $B$ be two non commuting elements of $\Sigma$. Let $X = \langle A, B \rangle$. Then we have the following:

(a) $A$ is regular on $\{C \in \Sigma \cap X \mid C \neq A\}$;

(b) if $C$ and $D$ are distinct elements of $\Sigma \cap X$, then $\langle C, D \rangle = X$;

(c) $X = \langle A, x \rangle$ for all $x \in X \setminus N_X(A)$.
(d) for each \( a \in A^2 \) we have \( C_X(a) = AZ(X) \);

(e) \( Z(X) \) has order at most 2;

(f) if \( a_1 \in A^2 \) and \( b_1, b_2 \in B^2 \) are non identity elements, then either \( A^{b_2a_1} = B \) or there are \( a_2, a_3 \in A \) and \( b_3 \in B \) with \( a_2b_1a_3 = b_2a_1b_3 \).

Proof. Most of these results are well known or follow easily from properties of \( SL_2(k) \); for a proof see [20]. We only prove the last assertion. Without loss we may assume that \( a_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \) and \( b_2 = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \), where \( s, t \) and \( u \) are nonzero elements in \( k \).

Then either \( us = -1 \) and \( A^{b_2a_1} = B \) or with \( a_2 = \begin{pmatrix} 1 & 0 \\ s(t-u) & 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 & 0 \\ 1+sus & 1 \end{pmatrix} \), and \( b_3 = \begin{pmatrix} 1 & t-u \\ 0 & 1 \end{pmatrix} \) we have \( a_2b_1a_3 = b_2a_1b_3 \). \( \Box \)

The following properties are straightforward consequences of the above; see [20, Section 2].

2.2 (a) If \( A, B \in \Sigma \) then \( A \cap B = 1 \) or \( A = B \).

(b) Let \( A, B \in \Sigma \), \( A \neq B \) and \( [A, B] = 1 \). If \( 1 \neq a \in A \) and \( 1 \neq b \in B \), then \( C_\Sigma(ab) = C_\Sigma(a) \cap C_\Sigma(b) \).

(c) \( Z_2(G) = Z(G) \).

Let \( L(\Sigma) \) be the set consisting of all sets \( \{ C \in \Sigma \mid C \subseteq \langle A, B \rangle \} \), where \( A \) and \( B \) are two non commuting elements in \( \Sigma \). Let \( \Pi \) be the pair \( (\Sigma, L(\Sigma)) \).

The group \( G \) acts on the space \( \Pi \) by conjugation. By (2.2)(c) the kernel of the action is contained in the center of \( G \). As we are mainly interested in the geometry \( \Pi \), we can and do assume, for the remainder of this section, that this center is trivial.

2.3 \( \Pi \) is a connected partial linear space.

Proof. It is a direct consequence of (2.1)(b) that two points are on at most one line and all lines contain at least two points. Hence \( \Pi \) is a partial linear space. Connectedness follows from the fact that \( \Sigma \) is a conjugacy class in \( \langle \Sigma \rangle \). \( \Box \)

If \( A \) and \( B \) are two non commuting points in \( \Sigma \), then we denote by \( AB \) the unique line in \( L(\Sigma) \) containing \( A \) and \( B \).

For any subset \( X \) of \( \Sigma \) we denote by \( \langle X \rangle \) the subgroup of \( G \) generated by \( X \) and by \( \langle X \rangle_\Pi \) the subspace of \( \Pi \) generated by \( X \). We notice that \( \{ C \in \Sigma \mid C \subseteq \langle X \rangle \} \) is not always the same as \( \langle X \rangle_\Pi \), it can also properly contain \( \langle X \rangle_\Pi \).
A triangle of $\Sigma$ is a triple $(A, B, C)$ of distinct elements of $\Sigma$ such that $[A, C] = 1$ and $[B, A] \neq [B, C]$. Here we follow the notation of Timmesfeld [20, 22].

2.4 Let $(A, B, C)$ be a triangle in $\Pi$, then the subspace $\langle A, B, C \rangle_\Pi$ of $\Pi$ is isomorphic to a dual affine plane.

Proof. Fix a point $D$ on the line $BC$ different from $B$ and $C$. Then $A$ is collinear to $D$. Let $\Delta$ be the set of points on the lines through $A$ meeting $BC$ or on the lines through $B$ meeting $AD$. We will show that $\Delta$ equals the subspace of $\Pi$ generated by $A$, $B$ and $C$.

By (2.1)(a) there are $a \in A$, $b \in B$ and $c \in C$ such that $D = C^b = B^c$, and $A^{ba} = B$. But then $B' = D^a = B^{ac}$ is a point on $AD$ commuting with $B$. By similar arguments we find that there is an element $t \in BB'$ with $A^t = C$. Fix such an element $t$.

We notice that for each $a_1 \in A^t$, the point $B^{a_1^{-1}a_1}$ is the unique point on the line $AB^{a_1^{-1}}$ commuting with $B$.

Indeed, suppose $a_1 \in A^t$. As above, choose $b_1 \in B$ and $c_1 \in C$ with $C^{b_1} = B^{c_1}$, and $A^{b_1c_1} = B$. Then $B^{a_1c_1}$ is a point commuting with $B$. Moreover, $B = B^t = A^{b_1c_1t} = A^{b_1t^{-1}a_1c_1} = C^{b_1a_1}$, so $b_1^{-1}$ and $a_1^{-1}$ are both elements in $C$ conjugating $C^b$ to $B$. By (2.1)(a), we have $a_1^{-1} = c_1^{-1}$ and $B^{a_1c_1^{-1}} = B^{a_1c_1}$ is the unique point on the line through $A$ and $B^{a_1}$ commuting with $B$.

Let $l$ be a line through $A$ meeting $BC$ in a point $E$. Let $H$ be a point on $l$ different from $A$. Let $b_1, b_2 \in B$ with $D = C^{b_1}$ and $E = C^{b_2}$ and $a_1 \in A$ with $H = E^{a_1}$. Now by (2.1)(f) either $B = A^{b_2a_1}$, or there are $b_3 \in B$ and $a_2, a_3 \in A$ with $b_2a_3b_3 = a_2b_1a_3$. In the first case we find that $H = C^{b_2a_1}$ commutes with $B = A^{b_2a_1}$. It also commutes with $B'$. Indeed, let $c_1 = a_1^{-t} \in C$ then $C^{b_2} = B^{c_1}$. Moreover, as $ca(c_1a_1)^{-1} = cc_1^{-1}a_1^{-1} = (aa_1)^{-1}t(aa_1^{-1})$, we see that $B$ commutes with $B^{c(a_1c_1)^{-1}}$, from which it easily follows that $H$ commutes with $B'$. In the second case we have $H^{b_3} = C^{b_2a_3b_3} = C^{a_2b_1a_3} = C^{b_1a_3} = D^{b_3}$, which implies that the lines $BH$ and $AD$ intersect. In particular $H$ is on a line through $B$ meeting $AD$.

This shows us that all points of $\Delta$ are on a line through $B$ meeting $AD$ or are in $B^\perp$. Moreover, $\Delta \cap B^\perp = \Delta \cap B_0^\perp$ for all $B_0 \in \Delta \cap B^\perp$. Thus $\Delta$ is $B$-invariant. By the same arguments we find that $\Delta$ consists of points on lines through $A$ meeting $BC$ and points in $A^t$, and thus is $A$-invariant.

Next we show that $\Delta$ is also $B'$-invariant. Suppose $P$ is a point in $\Delta$. Then by the above we find that $P$ is either on a line through $B$ meeting $AD$, in which case we clearly have $P^{b'} \in \Delta$ for all $b' \in B'$, or not. In the latter case $P$ commutes with $B$ and $B'$, as was shown above. This proves that $\Delta$ is $B'$-invariant.

Since $\langle A, B, B' \rangle = \langle A, B, C \rangle$, we find $\Delta$ to be $\langle A, B, C \rangle$-invariant. As $\Delta$ is connected, it contains the subspace of $\Pi$ generated by $A$, $B$ and $C$, and, by construction, we find it to be equal to this subspace.
As ∆ is connected, we find \langle A, B, C \rangle to act transitive (by conjugation) on the points of ∆. This group is also transitive on the lines of ∆. Indeed, suppose \( l \) is a line of ∆, we will show that \( l \) is in the \( \langle A, B, C \rangle \)-orbit of \( AB \). By transitivity on the points of ∆ we can assume \( A \in l \). So, either \( l = AB \) or it meets \( BC \) in a point \( E \). But then we can find a \( c \in C \) with \( E^c = B \) and thus \( l^c = AB \).

It remains to show that ∆ is a dual affine plane. Let \( l \) be a line in ∆, then up to conjugation with an element of \( \langle A, B, C \rangle \) we can assume that \( l = AB \).

It follows from the above description of ∆ that any point not on \( l \) is collinear with all but one points of \( l \). Thus we only have to check that any other line \( m \) in ∆ meets \( l \).

Let \( m \) be a line in ∆ different from \( l \). Without loss of generality we can assume that \( C \in m \). Suppose \( K \) is another point on \( m \). Then \( K \) is a point on a line through \( A \) meeting \( BC \) in a point, \( P \) say. Now there is a \( c \in C \) with \( P^c = B \), and therefore \( K^c \in A^cP^c \cap CK = l \cap m \), which proves that \( l \) and \( m \) have non trivial intersection. ∎

In a dual affine plane each point is contained in a unique maximal coclique of the collinearity graph of the plane. Such a coclique is called a transversal coclique. Any coclique of \( \Pi \) which is a transversal coclique of a subspace of \( \Pi \) isomorphic to a dual affine plane will be called a transversal coclique or just transversal of \( \Pi \).

2.5 Let \( (A, B, C) \) be a triangle in \( \Sigma \). Then \( N_{\langle A, B \rangle}(A) \) is transitive on the points different from \( A \) contained in the transversal coclique of \( \langle A, B, C \rangle_{\Pi} \) on \( A \).

Proof. Suppose \( D \) is a point in the transversal coclique of the dual affine \( \langle A, B, C \rangle_{\Pi} \) on \( A \) different from \( C \). Let \( l \) be a line of this dual affine plane on \( D \) and meeting the line \( AB \) in a point \( E \) different from \( B \). Then \( l \) meets also \( BC \) in a point, \( F \) say. Now let \( b \in B \) with \( C^b = F \). Then \( A^b \) is the unique point on \( AB \) not collinear with \( F \) and we can find an element \( e \in E \) with \( A^be = A \), i.e., \( be \in N_{\langle A, B \rangle}(A) \). But then \( F^e = C^be \) is the unique point on \( EF \) not collinear with \( A \) and thus equal to \( D \). This proves the lemma. ∎

2.6 Let \( (A, B, C) \) be a triangle in \( \Sigma \). Then for each \( 1 \neq a \in A \) there is a unique \( c \in C \) with \( [B, B^a] = 1 \). For all other \( c \in C \) there is a point \( D \) in the transversal coclique on \( A \) in the plane generated by the triangle containing an element \( d \in D \) with \( acd \in Z(\langle A, B, C \rangle) \).

Proof. Suppose \( a \in A \). On the line \( CB^a \) there is a unique point commuting with \( B \). As \( C \) is regular on the points on \( CB^a \setminus \{C\} \) there is a unique \( c \in C \) with \( B^ac \) commuting with \( B \).

Now suppose \( c \in C \) is an element such that \( B^ac \) does not commute with \( B \). Consider the line \( BB^ac \). This line intersects the transversal coclique on \( A \) in a point \( D \) say. There is a \( d \in D \) with \( B^{acd} = B \). In particular \( acd \) centralizes \( A, C \) and
normalizes $B$. Thus $T$ is transitive on the points of $AB$ different from $A$, we find that $acd$ normalizes all points on this line and thus centralizes all of them; see (2.1) and (2.2). Hence $acd \in Z((A, B, C))$.

2.7 Suppose $|k| \geq 4$. Let $T$ be a transversal coclique of $\Pi$ and $X$ a point not in $T$. If $|X^\perp \cap T| \geq 2$ then $T \subseteq X^\perp$.

Proof. Suppose $(A, B, C)$ is a triangle and $X$ a point not in the dual affine plane $\pi$ generated by $A, B$ and $C$ and commuting with $A$ and $C$. Let $D$ be a point on the transversal $T$ of $\pi$ on $A$ different from $A$ and $C$. Fix $d \in D$ not equal to 1. Then by the above lemma there are $a \in A$ and $c \in C$ with $acd \in Z((A, B, C))$. If $acd = 1$ then $X$ commutes with all points on $T$. For, by (2.5) we know that $N_{(A,B)}(A)$ is transitive on $T \setminus \{A\}$. Thus for any $E \in T \setminus \{A\}$ there is an $F \in T$ different from $E$ and $A$ such that $a'ef = 1$ for some non identity elements $a' \in A$, $e \in E$ and $f \in F$. Thus $ef = a'^{-1} \in A \in C_G(X)$. But then by (2.2)(c), we have $e \in C_G(X)$ and thus $[E, X] = 1$ by (2.1)(a).

Thus we can assume that $acd \neq 1$. By (2.5) there is an $n \in N_{(A,B)}(A)$ with $C^n = D$. Then 1 = $[acd, n] = [a, n][c, n][d, n] = [a, n]c^{-1}a^{-1}d^{-1}d^n$. Hence $X$ commutes with $1 \neq c[a, n]^{-1} = (c^a d^{-1}) d^n \in DD^n$. Now, by (2.1)(d) and (2.2)(a), either $X$ commutes with $D$ or $c^n = d$. Suppose $c^n = d$. Then $[a, n]c^{-1}d^n = 1$. If $D^n \neq C$, then $[a, n] \in A$, $c^{-1} \in C$ and $d^n \in D^n$ are three nontrivial elements whose product is 1. As before, but now with $d^n \in D^n$ instead of $d \in D$, we find that $X$ commutes with all points in $T$. Thus we may assume that $[a, n] \in C \cap A = 1$ and thus $n \in AZ((A, B))$; see (2.1).

Hence if $X$ does not commute with $D$ then $D \in C^{AZ((A, B))}$. Since the order of the center of $(A, B)$ is at most 2, $D$ is the unique element in $T$ not commuting with $X$. If $T$ contains more than 3 elements, then let $C'$ be an element in $T$ different from $A$ and not in $C^{AZ((A, B))}$. Applying the above to $A, C'$ and $D$ instead of $A, C$ and $D$ implies that $D$ commutes with $X$ as $D \notin C'^{AZ((A, B))}$. Thus, if $|k| \geq 4$, then $T \subseteq X^\perp$ leads to a contradiction proving the statement.

2.8 Suppose $A$ and $B$ are points of $\Sigma$ with $A^\perp \subseteq B^\perp$. Then $A^\perp = B^\perp$.

Proof. Suppose $A^\perp \subseteq B^\perp$. Since $\Sigma$ is a conjugacy class, there is a point $C \in \Sigma$ with $(A, C, B)$ being a triangle. Now there is an element $g \in \langle A, B, C \rangle$ with $A^g = B$ and hence $A^\perp \subseteq B^\perp \subseteq B^g\perp$.

If $k = F_3$ then (2.5) implies that we may take $g$ to have order 2, and the lemma follows immediately. If $|k| \geq 4$ we have by the above lemma $B^\perp = B^\perp \cap B^g\perp \subseteq A^\perp$ proving the lemma.

2.9 Suppose that $|k| > 3$. Then either for all $A \perp B \in \Sigma$ we have $A^\perp = B^\perp$ or $(\Sigma, \perp)$ is connected.
**Proof.** Suppose $|k| > 3$, and that there are $X \perp Y \in \Sigma$ with $X^\perp \neq Y^\perp$. Then we have to show that $(\Sigma, \perp)$ is connected. Assume $(\Sigma, \perp)$ not to be connected. Let $\Delta_0$ and $\Delta_1$ be two connected components of $(\Sigma, \perp)$. Let $A \neq C \in \Delta_0$, be commuting and $B \in \Delta_1$. Then $(A, B, C)$ is a triangle. By $I_{AB}$ we denote the set of connected components of $(\Sigma, \perp)$ meeting the line of $\Pi$ on $A$ and $B$ non trivially.

Inside $(A, B, C)$ it is straightforward to check that each point of $(A, B, C)_\Pi$ is contained in some element from $I_{AB}$ and that $I_{AB} = I_{CB}$. As $\Delta_0$ and $\Delta_1$ are connected components of $(\Sigma, \perp)$, this implies that $I_{AB} = I_{XY}$ for any $X \in \Delta_0$ and $Y \in \Delta_1$. In particular, $I_{AB}$ is invariant under the group $H := (\Delta_0, \Delta_1)$.

Fix a $B_0$ in the transversal of $(A, B, C)_\Pi$ on $B$, but distinct from $B$. The line $B_0C$ meets $AB$ in a point $D$. Let $b \in B$ with $A^b = D$ and $b_0 \in B_0$ with $D^{b_0} = C$, then $A^{b_0} = C$. Inside the dual affine plane on $A, B$ and $C$, we see that the action of $B$ on the different components in $I_{AB}$ is isomorphic with the action on the line $AB$. Moreover, as the latter action is regular, the non trivial element $bb_0$ fixes all elements in $I_{AB}$. So, $bb_0$ is in $N$, the normal subgroup of $H$ fixing all elements in $I_{AB}$. Moreover, since $\Delta_0$ is connected with respect to $\perp$, we find that $N$ is transitive on $\Delta_0$.

Since $\Sigma$ is a conjugacy class, there is an element $E$ in $A^\perp$ with $A^\perp \neq E^\perp$. Since $A^\perp \not\subseteq E^\perp$, see (2.8), we find an element $D \in \Delta_0$ not commuting with $A$. The above implies that there is an element $n \in N$ with $A^n = D$. Then $N \cap \langle A, A^n \rangle$ is a non trivial normal subgroup of $(A, A^n)$ different from the center of $(A, A^n)$. As $|k| > 3$, the group $(A, A^n)/Z((A, A^n))$ is simple. Hence $A \leq N$, which obviously is a contradiction. This proves $(\Sigma, \perp)$ to be connected.

The results from this section imply the following.

**2.10 Proposition.** If $|k| > 3$ and $\Sigma$ contains two elements $A, B$ with $A \perp B$ but $A^\perp \neq B^\perp$, then $\Pi(\Sigma)$ is a transvection geometry.

For $A, B \in \Sigma$ we write $A \equiv B$ if and only if $A^\perp = B^\perp$. The relation $\equiv$ is an equivalence relation. For each element $A \in \Sigma$ we denote by $[A]$ the $\equiv$-class of $A$. By $\Sigma/\equiv$ we denote the set of all such equivalence classes. The relation $\perp$ induces a relation on $\Sigma/\equiv$ by the $[A] \perp [B]$ if and only if $A \perp B$.

Now we can apply Theorem 1.3 and immediately find the following result.

**2.11 Theorem.** Suppose $k$ is a field with at least 4 elements. Let $G$ be a group generated by a class $\Sigma$ of $k$-transvection subgroups containing two commuting elements $A, B$ with $A^\perp \neq B^\perp$. Then the graph $(\Sigma/\equiv, \perp)$ is a non degenerate polar graph of rank at least 2. The elements of $A \in \Sigma$ induce polar transvections on the graph $(\Sigma/\equiv, \perp)$.

This theorem together with the next result implies Theorem 1.4. Notice that the result does not require the field to have more than three elements.

10
Let \( k \) be a field. Suppose \( \Sigma \) is a class of \( k \)-transvection subgroups generating the group \( G \) with \( (\Sigma/\equiv, \perp) \) a nondegenerate polar graph of rank at least 2. If \( \equiv \) is non trivial, then \( G \) contains a non trivial normal subgroup \( N \) with \( [N, G] \) Abelian.

**Proof.** Suppose \( \equiv \) is non trivial. Let \( N \) be the kernel of the action of \( G \) on the \( \equiv \)-classes. Let \( A \in \Sigma \) and \( C \in \langle A \rangle \) different from \( A \). Suppose \( B \in \Sigma \) is not commuting with \( A \). So, \((A, B, C)\) is a triangle. Fix an element \( a \in A^\circ \). Then there is an element \( c \in C \) with \( B^{ac} \) commuting with \( B \); see (2.6). The element \( B^{ac} \) is in \( [B] \) as we can easily check inside \( \langle A, B, C \rangle \). So, \( ac \) fixes all \( \equiv \)-classes in \( [A] \perp \) as well as \( [B] \). But then \( ac \) fixes all points of the polar graph \((\Sigma/\equiv, \perp)\). So \( 1 \not= ac \in N \).

Next we will show that \([N, G]\) is Abelian. Let \( A, B \in \Sigma \). As \([N, A]\) and \([N, B]\) are normal in \( N \), the commutator \([N, A] \cap [N, B]\) is contained in \([N, A] \cap [N, B]\). However, as we have seen above, \([N, A] \cap [N, B]\) is centralized by both \( \langle C_\Sigma(A) \rangle \) and by \( \langle C_\Sigma(B) \rangle \) and therefore by \( \langle C_\Sigma(A), C_\Sigma(B) \rangle \). But \( \langle C_\Sigma(A), C_\Sigma(B) \rangle \) equals \( G \), as can be easily checked using (2.11). This implies that that \([N, A] \cap [N, B]\) \leq Z(G) which we assume to be trivial. As \( G \) is generated by \( \Sigma \), we find \([N, G]\) to be Abelian. \( \square \)

3 The geometry of \( \mathbb{F}_3 \)-transvection groups

In this section we prove Theorem 1.6. Let \( G \) be a group and \( \Sigma \) a class of \( \mathbb{F}_3 \)-transvection subgroups of \( G \). As before, define \( L(\Sigma) \) to be the set of all sets \( \{C \in \Sigma | C \subseteq \langle A, B \rangle \} \) where \( A \) and \( B \) are two non commuting elements of \( \Sigma \). Denote by \( \Pi = \Pi(\Sigma) \) the partial linear space \((\Sigma, L(\Sigma))\).

3.1 We start with a list of some groups generated by 3 elements of order 3, such that the conjugates of the subgroups generated by these elements form a conjugacy class of \( \mathbb{F}_3 \)-transvection subgroups in the group they generate. Moreover, we study the geometry induced on this conjugacy class.

Let \( a \) and \( b \) be two distinct conjugated elements of order 3 in \( SL_2(3) \). Then \( a \) and \( b \) generate \( SL_2(3) \) and satisfy the relations

\[ a^3 = b^3 = 1; \quad aba = bab. \]

In fact these relations define \( SL_2(3) \). This group has a center \( Z \) of order 2 generated by \( \langle ab^{-1} \rangle^2 \). So the group \( Alt_4 \simeq SL_2(3)/Z \) can be defined by the relations

\[ a^3 = b^3 = 1; \; aba = bab; \; a^{-1}b = b^{-1}a. \]

Now suppose \( a, b \) and \( c \) are three elements of order 3 generating a group \( G \). Following [2] we write \( x \sim y \) if \( x \) and \( y \) satisfy the relations \( x^3 = y^3 = 1, \ xyx = yxy \).

Using the results of Section 3 of [2] or coset enumeration with the help of GAP, [15], we can list the possibilities for \( G \) in the case that \( \langle a \rangle, \langle b \rangle \) and \( \langle c \rangle \) are in a single
class $\Sigma$ of $\mathbb{F}_3$-transvection subgroups of $G$. We only list those cases where one needs at least 3 elements of this class to generate $G$. Without loss we can assume $a \sim b \sim c$. Moreover, We distinguish the following cases.

(a) $ac = ca$. Then $G \cong 3^{(1+)^2} : \text{SL}_2(3)$ and $\Pi$ is a dual affine plane.

(b) $c \sim a, b^a$ and $a^b$. Then $G \cong 2^{n+4} : 3$ where $0 \leq n \leq 4$ and $\Pi$ is isomorphic to an affine plane. If two points on a line of $\Pi$ generate an $\text{SL}_2(3)$ then all lines parallel to that line also generate an $\text{SL}_2(3)$ and they all have the same central involution (called the center of the line). The normal subgroup of order $2^n$ of $G$ is elementary Abelian and is generated by the central involutions of the lines.

(c) $c \sim a, c^{-1} \sim b^a$ and $c^{-1} \sim a^b$. Then $G \cong \text{PSU}_3(3)$ and $\Pi$ is isomorphic to the classical unital related to $\text{PSU}_3(3)$.

(d) $c \sim a, c^{-1} \sim b^a$ and $c \sim a^b$. Then $G$ is the trivial group.

This has the following consequence.

3.2 Let $\Sigma$ be a class of $\mathbb{F}_3$-transvection subgroups. Then $\Pi(\Sigma)$ is a connected partial linear space of order 3 such that all planes are isomorphic to a dual affine plane, an affine plane or a classical unital.

The subgroup of $G = \langle \Sigma \rangle$ generated by the points of a plane $\pi$ of $\Pi(\Sigma)$ is isomorphic to $3^{(1+)2} : \text{SL}_2(3)$ when $\pi$ is a dual affine plane, to $2^{n+4} : 3$, where $0 \leq n \leq 4$, and $\Pi$ when $\pi$ an affine plane and to $\text{PSU}_3(3)$ when $\pi$ is a classical unital.

Proof. $\Pi(\Sigma)$ is connected since $\Sigma$ is a conjugacy class in $G = \langle \Sigma \rangle$.

Suppose $A, B$ and $C$ are elements from $\Sigma$ such that $AB$ and $BC$ are lines in $\Pi(\Sigma)$ generating a plane $\pi$ in $\Pi(\Sigma)$.

If $C$ is not collinear to some point on $AB$, then, without loss of generality, we may assume that $A \perp C$ and we can choose elements $a \in A^2, b \in B^2$ and $c \in C^5$ satisfying (3.1)(a). So, $\pi$ is a dual affine plane and $\langle \pi \rangle \cong 3^{(1+)^2} : \text{SL}_2(3)$.

If $C$ is collinear to all points of $AB$, we can choose elements $a \in A^2, b \in B^2$ with $a \sim b$. If $c$ is in relation $\sim$ with one or three of the elements $a, b, a^b$ and $b^a$, then (up to permutation of $a, b, a^b$ and $b^a$ and replacing $c$ by its inverse, if necessary) we are in case (3.1)(d), and $G$ is trivial. If $c$ is in relation $\sim$ with two or four elements from $a, b, a^b$ and $b^a$, then (again up to permutation of $a, b, a^b$ and $b^a$) we are in case (3.1)(c) or (3.1)(b), respectively. Thus $\pi$ is a unital or an affine plane and $\langle \pi \rangle$ is as described.

The following result is 3.6 and 3.8 of Aschbacher and Hall [2].

3.3 Let $\Sigma$ be a class of $\mathbb{F}_3$-transvection subgroups. A unital in $\Pi(\Sigma)$ has the property that any point outside the unital is collinear with at most all but one of the points of the unital. If $\Pi(\Sigma)$ contains a unital then all planes are either dual affine or unitals.
First we consider the case that there are no commuting elements in $\Sigma$. The following theorem is concerned with that situation and gives an answer to a question posed by Timmesfeld [20, Section 1].

3.4 Theorem. Let $G$ be a group generated by a class $\Sigma$ of $F_3$-transvection subgroups. If no two distinct elements of $\Sigma$ commute, then either $G$ is isomorphic to $PSU_3(3)$ or there is a chain $1 \leq Z(G) \leq K \leq G$ with $Z(G)$ and $K/Z(G)$ elementary Abelian 2-groups and $G/K$ of order 3.

Proof. Consider $\Pi = (\Sigma, L(\Sigma))$. Then by the above proposition, $\Pi$ is either a unital and $G$ is isomorphic to $PSU_3(3)$ or all planes in $\Pi$ are affine. But in the latter case a theorem of Buekenhout [3] implies that $\Pi$ is an affine space. So in that case, we find that the center of each line is contained in $Z(G)$. Let $H$ be the group generated by all these centers of the lines, then $H$ is an elementary Abelian 2-group.

Now consider $G/H$. Then any element of order 2 in the subgroup of $G/H$ generated by the 4 points on a line defines a translation on $\Pi$. Modulo the group $T$ generated by the translations, which is also an elementary Abelian 2-group, all elements of $\Sigma$ are the same. Denote by $K$ the preimage of $T$ in $G$. Then $1 \leq H \leq K \leq G$, with $H = Z(G)$ and $T = K/H$ being elementary Abelian 2-groups and $G/K$ a group of order 3. Thus we have proved the theorem. □

The next result implies Theorem 1.6.

3.5 Theorem. Suppose $\Sigma$ is a class of $F_3$-transvection subgroups generating a group $G$. If $G$ contains no non trivial nilpotent normal subgroup, then up to isomorphism we have one of the following:

(a) $\Pi(\Sigma)$ is the geometry of singular points and hyperbolic lines of a non degenerate symplectic $F_3$-space $(V, f)$ of dimension at least 4; $\Sigma$ is the class of transvection subgroups in $PSp(V, f)$.

(b) $\Pi(\Sigma)$ is the geometry of singular points and hyperbolic lines of some non degenerate unitary $F_2$-space $(V, h)$ of dimension at least 3; $\Sigma$ is the class of transvection subgroups in $PSU(V, h)$.

(c) $\Pi(\Sigma)$ is the geometry of non singular points and tangent lines of some non degenerate unitary $F_2$-space $(V, h)$ of dimension at least 4; $\Sigma$ is the class of triflection subgroups in $PU(V, h)$.

Proof. Consider the partial linear space $\Pi = (\Sigma, L(\Sigma))$. We distinguish three different cases:

1. All planes are linear.
If this space contains only linear planes, then we are in the case covered by Theorem 3.4 and we only find the group $PSU_3(3)$. 13
2. The space $\Pi$ contains dual affine planes but no affine planes.

Suppose there are no affine planes in $\Pi$. Then planes in $\Pi$ are dual affine or classical unitals. Such geometries have been considered in [12]. In particular, it follows from Theorem 2.11 and Section 3 of [12] that $(P/\equiv, \perp)$ is either a polar graph of a non-degenerate symplectic or unitary polar space over the field $\mathbb{F}_3$ or $\mathbb{F}_{3^2}$, respectively, or $(P, \perp)$ is a union of cliques.

In the first situation (2.12) implies that $\equiv$ is trivial and it readily follows that we are in case (a) or (b) of the conclusion of the theorem.

Suppose we are in the latter case, i.e., $(P, \perp)$ is a union of cliques. In this case the geometry $\Pi/\equiv$ with as points the $\equiv$-classes and as lines the sets of $\equiv$-classes meeting a line of $\Pi$ is either just a line or a unital, as follows from (3.3).

Suppose $\equiv$ is nontrivial. Let $N$ be the normal subgroup of $G$ consisting of those elements that act trivially on the set of $\equiv$-equivalence classes. Let $A_1 \neq A$ be $\equiv$-equivalent elements of $\Sigma$. Fix elements $a \in A^2$ and $a_1 \in A_1^2$ such that $B^{aa_1} \perp B$. Then, as can be checked inside the dual affine plane on $A, B$ and $A_1$, the element $aa_1$ acts trivial on the set of $\equiv$-classes meeting the line $AB$ nontrivially. But this also implies that $aa_1$ acts trivially on the unital induced on $\Pi/\equiv$. So, $1 \neq aa_1 \in N$.

The second derived subgroup of $[N, G]$ is generated by subgroups of the form $[[[N, A], [N, B]], [N, C]]$, where $A, B, C \in \Sigma$. As for each $D \in \Sigma$ the subgroup $[N, D]$ is normal in $[N, G]$, we have

$$[[N, A], [N, B]] \leq [N, A] \cap [N, B]$$

and

$$[[[N, A], [N, B]], [N, C]] \leq [N, A] \cap [N, B] \cap [N, C].$$

Thus the subgroup $[[[N, A], [N, B]], [N, C]]$ is nontrivial only if $A, B$ and $C$ generate a unital. But in that case $[[[N, A], [N, B]], [N, C]]$ is centralized by $\langle C_\Sigma(A), C_\Sigma(B), C_\Sigma(C) \rangle = G$. This proves $N$ to be nilpotent, contradicting the assumptions of the theorem.

So we can assume $\equiv$ to be trivial. This implies that $\Pi$ is either a line, which leads to a contradiction, since then $G \cong (P)\text{SL}_2(3)$ is solvable, or $\Pi$ is a unital and $G \cong \text{PSU}_3(3)$ as in conclusion (b).

This finishes case 2.

3. The space $\Pi$ contains affine and dual affine planes.

Now assume that $\Pi$ does contain affine planes. Then by (3.3) there are no unitals in $\Pi$ and $\Pi$ is a generalized Fischer space as considered in [6, 8, 10].

Let $x$ and $y$ be points, then we write $x \approx y$ if and only if $x^+ \setminus \{x\} = y^+ \setminus \{y\}$. The relation $\approx$ is an equivalence relation. To be able to apply the results of [6, 8, 10] and to conclude that $\Pi$ is as described in part (c) of the theorem we have to prove $\approx$ to be trivial.

Suppose there are distinct points $x$ and $y$ with $x \approx y$. Let $N$ be the kernel of the action of $G$ on the set of $\approx$-equivalence classes. We will prove $N$ to be a nontrivial nilpotent subgroup of $G$. 

14
Let \( K \) be the non trivial maximal normal 2-group in the group \( \langle x, y \rangle \). It follows from [6, 8, 10] that for each \( z \) collinear to \( x \) the plane generated by \( xy \) and \( z \) is affine, and the points on lines parallel to \( xy \) in this plane are also \( \approx \)-equivalent. In particular, \( K \) fixes these lines. Hence \( K \) acts trivial on the set of \( \approx \)-equivalence classes and is contained in \( N \).

Let \( A, B \in \Sigma \). As before we have that \( [N, A] \) is normal in \( N \) which implies \( [[N, A], [N, B]] \) to be contained in \( [N, A] \cap [N, B] \). Clearly \( [[N, A], [N, B]] = 1 \), if \( [A, B] = 1 \) and \( B \neq A \). If \( A \) and \( B \) are collinear, and \( C \in \Sigma \), then within the subgroup \( \langle A, B, C \rangle \) of \( G \) we find \( [[N, A], [N, B]] \) to be centralized by \( C \); see (3.2). So, \( [[N, A], [N, B]] \) is central in \( G \).

The above implies \( [[N, G], [N, G]] \) to be central in \( G \) and \( N \) to be nilpotent, which contradicts the hypothesis of the theorem. Thus we can conclude that \( \approx \) is trivial, and we have finished the last case. In particular, in this case we find \( \Pi \) to be the geometry of tangents of a unitary polar space over the field \( \mathbb{F}_2 \) and \( \Sigma \) can be identified with the class of triflection subgroups in the corresponding unitary group.

\[ \square \]

4 From transvection geometries to groups

In this last section we will show how to recover a class of \( k \)-transvection subgroups from a transvection geometry. We continue with the notation of Section 2.

Let \( S \) be a polar space with point set \( P \). For a point \( p \in P \) the set \( p^\perp \) denotes the set of all points of \( S \) collinear (in \( S \)) to \( p \). A hyperbolic line of \( S \) is a set of points of the form \( \{ r \in S \mid r \in r^\perp \text{ for all } x \in p^\perp \cap q^\perp \} \), where \( p \) and \( q \) are two noncollinear points of \( S \).

Assume that \( \Pi = (P, L) \) is a transvection geometry satisfying the hypothesis of Theorem 1.5. By Theorem 1.3 we find the graph \( (P/\equiv, \perp) \) to be the polar graph of a non degenerate polar space \( S \). This polar space has rank at least 3 as follows immediately from condition (b) of Theorem 1.5.

We denote by \( \overline{\Pi} \) the geometry with as point set \( P/\equiv \) and as lines the sets \( \{ [x] \mid x \in l \} \), where \( l \in L \).

4.1 The lines of \( \overline{\Pi} \) are the hyperbolic lines of \( S \).

\textbf{Proof.} Let \( l \in L \). Then clearly \( \{ [x] \mid x \in l \} \) is contained in a hyperbolic line of \( S \). Denote this hyperbolic line by \( h \) So it remains to prove that \( h \) is contained in \( \{ [x] \mid x \in l \} \).

Suppose not and let \( y \in P \) with \( [y] \in h \), but not in \( \{ [x] \mid x \in l \} \). The subspace \( \langle l, y \rangle_{\Pi} \) is then a linear plane. Now by assumption (a) of (1.5), there is a point \( u \in P \) with \( u^\perp \cap \langle l, y \rangle = l \). But then \( \{ [x] \mid x \in l \} \subseteq [u]^\perp \) but \( [y] \notin [u]^\perp \), which is a contradiction. \( \square \)

15
4.2 Proposition. There is a class of \( k \)-transvection subgroups \( \Sigma \) in a group \( \overline{G} = \langle \Sigma \rangle \) with \( \Pi(\overline{\Sigma}) \cong \Pi \).

Proof. Any non degenerate polar space of rank at least 3 with hyperbolic lines with more than 3 points, is either classical (i.e., defined by some pseudoquadratic or alternating form) or a rank 3 polar space of type \( E_6 \) over a field \( k \); see [5, Chapter 12, 3.34]. (Notice that the line Grassmannian of a 3-dimensional projective space has thin hyperbolic lines.) For each point \( p \) of \( S \) let \( T_p \) be the subgroup of \( \text{Aut}(S) \) generated by all polar transvections of \( S \) with center \( p \). As hyperbolic lines of \( S \) are thick, Theorem 3.3 of [5, Chapter 12] implies that \( T_p \) is nontrivial. By \( \overline{\Sigma} \) denote the set of all subgroups \( T_p \) with \( p \) a point of \( S \). If \( S \) is a polar space of type \( E_6 \) over a field \( k \), then \( \overline{\Sigma} \) is a class of \( k \)-transvection subgroups, as follows from [22]. If \( S \) is a classical polar space, then [12, Section 2] reveals that the hypothesis of Theorem 1.5 implies that there is a field \( k \) such that \( \overline{\Sigma} \) is a class of \( k \)-transvection subgroups. Using [12, Section 2] (or [22]) we easily find \( \Pi(\overline{\Sigma}) \) to be isomorphic to \( \Pi \). \( \square \)

We will now lift the class \( \overline{\Sigma} \) to a class of \( k \)-transvection subgroups in the automorphism group of \( \Pi \), thereby using some ideas of Jonathan Hall; see [16]. We identify \( \Pi \) with \( \Pi(\overline{\Sigma}) \).

Let \( \overline{A} \in \overline{\Sigma} \) with center \( \overline{A} \). For each element \( \overline{a} \in \overline{A} \) and point \( x \in P \) with \( [x] = \overline{A} \), we define \( a_x : P \to P \) as follows.

\[
y^{a_x} = y \quad \text{if} \quad y \in x^\perp \\
y = z \quad \text{if} \quad y \notin x^\perp,
\]

where \( z \) is the intersection point of \( [x, y]_\Pi \) and \( [y]_\overline{\Pi} \).

Before showing \( a_x \) to be an automorphism of \( \Pi \), we need the following auxiliary result.

4.3 For any dual affine plane \( \pi \) of \( \Pi \), the subspace \( \langle [y] \mid y \in \pi \rangle_\Pi \) of \( \Pi \) is isomorphic to the geometry of points and lines in a projective space missing a codimension 2 subspace.

Proof. Suppose \( \pi \) is a dual affine plane in \( \Pi \). Then any two intersecting lines in the subspace \( \langle [y] \mid y \in \pi \rangle_\Pi \) of \( \Pi \) generate a dual affine plane. By [7], this subspace of \( \Pi \) is isomorphic to the geometry of points and lines in a projective space missing a codimension 2 subspace. \( \square \)

4.4 The map \( a_x \) is an automorphism of \( \Pi \).

Proof. The map \( a_x \) is clearly a permutation of the point set of \( \Pi \). So, we only have to check that it maps lines to lines. But first notice that \( a_x \) respects non collinearity, and thus also collinearity, of two points, as \( \overline{a} \) does.
Consider a line \( l \in L \). If \( l \) is a line on \( x \) or contained in \( x^\perp \), then \( l \) is \( a_x \) invariant and mapped to itself.

If \( l \) meets \( x^\perp \) in a unique point different from \( x \), then \( \langle x, l \rangle_{\Pi} \) is a dual affine plane. Consider the subspace \( \Delta \) of \( \Pi \) generated by \( \{ [y] \mid y \in \langle x, l \rangle_{\Pi} \} \). Inside \( \Delta \) we see that \( a_x \) is an automorphism and thus maps \( l \) to a line; see (4.3).

If \( l \) does not meet \( x^\perp \), then \( \pi = \langle x, l \rangle_{\Pi} \) is a linear plane. Let \( u, v, w \) be three points of \( l \). Then \( u^ax, v^ax \) and \( w^ax \) are three points in \( \pi \). Let \( m \) be the line through \( u^ax \) and \( v^ax \). We will show that \( w^ax \) is a point on \( m \).

By condition (b) of (1.5), there is a point \( z \in P \) such that \( z^\perp \cap \pi \) contains \( l \) but not \( x \). Then \( z^ax \) is a point of \( [z]^{\Pi} \) and thus not collinear to \( u^ax \) and \( v^ax \). In particular, \((z^ax)^\perp \cap \pi \) contains \( m \). Since \( x \notin (z^ax)^\perp \), condition (1.5)(a) implies that \((z^ax)^\perp \cap \pi = m \)

However, as \( z \perp w \), we also have \( z^ax \perp w^ax \), proving that \( w^ax \) has to lie on \( m \). Thus \( a_x \) is indeed an automorphism of \( \Pi \).

For each point \( x \in P \), let \( A_x \) be the subgroup of the automorphism group of \( \Pi \) generated by the elements \( a_x \), where \( \pi \in \overline{A} = [x] \). By \( \Sigma \) we denote the set of all subgroups \( A_x \) where \( x \in P \).

4.5 The set \( \Sigma \) is a class of \( k \)-transvection subgroups in \( G := \langle \Sigma \rangle \). The geometry \( \Pi(\Sigma) \) is isomorphic to \( \Pi \).

**Proof.** Consider the action of \( G \) on \( \Pi \), and denote its kernel by \( N \). Let \( \overline{A} \in \Sigma \) and \( A = A_x \) for some point \( x \in [\overline{A}] \). Consider \( n \in A \cap N \). If \( z \in x^\perp \), then \( z^n = z \).

If \( z \notin x^\perp \), then \( x, z \) are contained in some \( A \)-invariant subspace of \( \Pi \) isomorphic to the geometry of points and lines of a (Desarguesian) projective space missing a codimension 2 subspace. Inside this subspace we see that \( z^n \) equals \( z \). Thus \( A \cap N = \{1\} \). Moreover, \( AN/N \simeq \overline{A} \). So, \( A \simeq \overline{A} \) is an elementary Abelian group.

Now suppose \( \overline{B} \in \Sigma \) and \( B := B_y \) for some point \( y \in P \) with \([y] = \overline{B} \).

If \([\overline{A}, \overline{B}] = 1 \), then consider \( n \in N \cap [A, B] \). If \( z \in x^\perp \cup y^\perp \) it is readily seen that \( z^n = z \). If \( z \) is collinear to both \( x \) and \( y \), then \( x, y, z \) are contained in a subspace isomorphic to the geometry of points and lines in a projective space missing a codimension 2 subspace; see (4.3). Inside this subspace we find \( z^n = z \). Thus \( N \cap [A, B] = 1 \) and hence \([A, B] = 1 \).

If \([\overline{A}, \overline{B}] \simeq (P)SL_2(k) \), then let \( n \in N \cap (A, B) \). Let \( l \) be the line through \( x \) and \( y \) and suppose \( z \) is a point in \( P \). If \( z \in x^\perp \cup y^\perp \) or \( \langle x, y, z \rangle_{\Pi} \) is contained in a dual affine plane, then \( z^n = z \) as can be checked with a subspace of \( \Pi \) isomorphic to the geometry of points and lines in a projective space missing a codimension 2 subspace; see (4.3). Thus assume that \( \langle x, y, z \rangle \) is a linear plane, \( \pi \) say. Clearly \( n \) leaves \( \pi \) invariant. So \( z^n \in [z] \setminus \pi = \{z\} \). Hence \( n = 1 \). As a result we obtain that \([A, B] \simeq [\overline{A}, \overline{B}] \simeq (P)SL_2(k) \). By connectivity of \( \Pi \), this proves \( \Sigma \) to be a class of \( k \)-transvection subgroups in \( G \). The result follows immediately. \( \square \)

The above result implies Theorem 1.5.
References


18


